Policy Persistence and Drift in Organizations*

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Abstract

This paper models the evolution of organizations that allow free entry and exit of members, such as cities and trade unions. In each period current members choose a policy for the organization. Policy changes attract newcomers and drive away dissatisfied members, altering the set of future policymakers. The resulting feedback effects take the organization down a "slippery slope" that converges to a myopically stable policy, even if the agents are forward-looking, but convergence becomes slower the more patient they are. The model yields a tractable characterization of the steady state and the transition dynamics. The analysis is also extended to situations in which the organization can exclude members, such as enfranchisement and immigration.

Keywords: dynamic policy choice, median voter, slippery slope, endogenous population, transition dynamics

1 Introduction

This paper studies the dynamic behavior of organizations that are member-owned—that is, whose members choose policies through a collective decision-making process—

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and allow for the free entry and exit of members. In this context, policy and membership decisions affect one another: different policies appeal to or drive away different prospective members, and different groups make different choices when in charge of the organization. As a result, the policy path may drift over time: an initial policy may attract a set of members wanting a different policy, which in turn attracts other agents, and so on. It may also exhibit path-dependence: two organizations with identical fundamentals but different initial policies may exhibit divergent behavior in the long run.

A prominent example where these issues arise is that of cities or localities. Conceptualize each city as an organization and its inhabitants as members. Cities allow people to move in and out freely, and their inhabitants vote for local authorities who implement policies, such as the level of property taxes, the funding of public schools and housing regulations. The interplay between policy changes and migration can lead to demographic and socioeconomic shifts interpreted as urban decay, revitalization, and gentrification (Marcuse 1986; Vigdor 2010).

The relationship between local taxes and migration has been studied since Tiebout (1956), which considers a population able to move across a collection of communities with fixed policies. Epple and Romer (1991) allow for redistributive policies and location decisions to respond to one another, but they study the problem in static equilibrium, i.e., under the assumption that any temporary imbalance between the people living in a community and the policies they want has already resolved itself.

A natural follow-up question to this literature is whether, in a dynamic setting, communities will converge to a static equilibrium quickly, slowly or not at all. The main reason convergence may fail to obtain is a fear of *slippery slopes*. Here is a simple example: suppose a community with a local tax rate $x_0 = 0.2$ attracts a population whose median voter, m_0, prefers a tax rate $x_1 = 0.18$. Lowering the tax rate to x_1 would attract a different population whose median voter, m_1, has bliss point $x_2 = 0.16$. In turn, the tax rate x_2 would beget a median voter wanting a tax rate $x_3 = 0.15$, and so on. If agents vote myopically, the tax rate will quickly move not to x_1 but to a much lower steady state, say $x_\infty = 0.1$. Foreseeing this, m_0 might prefer not to change the tax rate after all.

This paper shows that, in a dynamic model where both policies and membership are determined endogenously, communities *will*, in fact, converge to a steady state, and all steady states are myopically stable independently of the agents' discount factor. However, dynamic concerns induce agents to make smaller policy changes in each period than their myopic preferences would dictate. In particular, when the median voter's bliss point is closer to the current policy than to the steady state, convergence is slow—that is, as agents become arbitrarily patient, policy changes in each period become arbitrarily small. Thus communities observed in the world at any given time may well fail to be in static equilibrium, and predicting their future behavior requires an understanding of their transition dynamics. The model also yields a tractable characterization of these dynamics, which allows us to describe the equilibrium speed of policy change in terms of the strength of the myopic incentives for change and the degree of expected disagreement with future pivotal agents.

The location of steady states is characterized as a function of the distribution of preferences. In general, policy drift leads organizations towards peaks of the distribution of policy bliss points, which favors centrism if said distribution is unimodal and symmetric. However, a pocket of agents concentrated at an extreme can also support a steady state. When there are multiple steady states, which one the organization converges to depends on its initial policy (i.e., there is path-dependence). Extreme steady states are more likely if agents' willingness to join is asymmetric (that is, extremists are more willing to join a moderate organization than vice versa). Relative to a setting with a fixed population, the location of steady states is more sensitive to the distribution of preferences: small changes in the distribution can result in arbitrarily large changes to the long-run policy.

This paper is connected to several strands of literature. First, as noted previously, it can be seen as a study of dynamic Tiebout competition. There is a large literature on the Tiebout hypothesis (see Cremer and Pestieau (2004) for a review), but most papers in it assume that policies and location decisions must be in static equilibrium, and hence are silent on the transition dynamics that this paper focuses on.

Second, the model can be applied to other organizations with open membership, such as trade unions, nonprofits, sports clubs and religious communities, and it is therefore relevant to existing work about such organizations. For instance, Grossman (1984) explains why increased international competition may not decrease wages in a unionized sector: layoffs selectively affect less senior workers, so the median voter within the union becomes more senior—hence more securely employed, and prone to making more aggressive wage demands. As in the Tiebout literature, Grossman (1984) assumes that policy and membership are always in static equilibrium, i.e., that they adjust immediately after an external shock; this paper can be seen as providing a model of the transition dynamics.

Finally, the paper makes several contributions to a growing literature on dynamic political decision-making (Roberts 2015; Acemoglu, Egorov and Sonin 2015; Bai and Lagunoff 2011). Most papers in this literature study organizations which can strategically restrict the entry of newcomers, remove existing members, or deny them political power (relevant applications include enfranchisement and immigration). Despite the apparent substantive differences between this setting and mine, my results readily extend to this context. The reason is that both types of models are driven by the same tension, namely, that policies and decision-making power are coupled in a rigid manner, so agents cannot choose their ideal policy without relinquishing control over future decisions.

There are two main branches in this literature. The first one (Roberts 2015; Acemoglu, Egorov and Sonin 2008, 2012, 2015) assumes a fixed, finite policy space and obtains the result that, when agents are patient enough, convergence to a steady state is “fast”, and steady states may not be myopically stable. The set of steady states can be found by means of a recursive algorithm, but not described explicitly, and it is sensitive to the set of feasible policies. What I show is that, if a continuous policy space is assumed, these results are overturned: all steady states are myopically stable, and when agents are patient, there is slow convergence which can be characterized explicitly (in some cases, in closed form).

The second branch (Jack and Lagunoff, 2006; Bai and Lagunoff, 2011) considers continuous policy spaces and obtains some important results related to the ones in this paper; in particular, Bai and Lagunoff (2011) show that, in their model, the steady states of “smooth” equilibria are stable under the assumption of a fixed decision-maker (in our setting, this is equivalent to myopic stability). However, they do not provide a general characterization of which steady state the model will converge to, nor of the transition dynamics. Moreover, their analysis applies only to smooth equilibria, which do not exist generically. In contrast, I derive results that apply either to all equilibria or to classes of equilibria for which I can provide existence conditions.

On a technical note, the present paper is also the first in this literature to tractably analyze a setting that violates the single-crossing assumption on preferences—a necessary complication in a context with free entry and exit, stemming from the fact that agents unhappy with the chosen policy can cut their losses by leaving the organization.

The paper is structured as follows. Section 2 presents the model. Section 3 proves some fundamental properties of all equilibria and characterizes the organization's policy in the long run. Section 4 characterizes the transition dynamics. Section 5 adapts the results to a setting without free entry and exit. Section 6 discusses some implications of the results and revisits their relationship with the existing literature. Section 7 is a conclusion. All the proofs can be found in the Appendices.

2 The Model

There is an organization (henceforth, a club) existing in discrete time $t = 0, 1, \ldots$ and a unit mass of agents distributed according to a continuous density f with support $[-1, 1]$. We refer to an agent's position α in the interval $[-1, 1]$ as her type. All agents are potential members of the club.

At each integer time $t \geq 1$, two events take place. First there is a *voting stage*, in which a set of *incumbent members* $I_{t-1} \subseteq [-1, 1]$ vote on a *policy* $x_t \in [-1, 1]$ to be implemented during the period $[t, t+1)$. Immediately after, in the *membership stage*, all agents observe x_t and decide whether to be members during the upcoming period $[t, t+1)$. Agents can freely enter and leave the club as many times as desired at no cost. The set of agents who choose to be members at time t constitutes I_t, the set of incumbent members at the $t+1$ voting stage.[1] At $t = 0$ the game starts with a membership stage; the club's initial policy x_0 is exogenously given.

The essential feature of this setup is that membership affects both an agent's utility and her right to vote. Agents will decide whether to be in the club based on their private payoffs, since the impact of any individual agent's vote on future policies is nil, but aggregate membership decisions will influence future policies.

Preferences

An agent α has utility

$$U_\alpha\left((x_t)_t, I_\alpha\right) = \sum_{t=0}^{\infty} \delta^t I_{\alpha t} u_\alpha(x_t),$$

[1]The assumption that agents vote the period *after* joining rules out equilibria in which agents who dislike the current policy might join because they expect the policy to immediately change to their liking. Equivalent results would be obtained by assuming that agents can enter or leave at any time $t \in \mathbb{R}_{\geq 0}$ but only gain voting rights after being members for a short time $\varepsilon \in (0, 1)$.

where $I_{\alpha t} = \mathbb{1}_{\{\alpha \in I_t\}}$ denotes whether α is a member at time t. In other words, the agent can obtain a payoff $u_\alpha(x_t)$ from being a member of the club, or a payoff of zero from remaining an outsider. $\delta \in (0,1)$ is a common discount factor. We make the following assumptions on u.

A1 $u_\alpha(x) : [-1,1]^2 \to \mathbb{R}$ is C^2.

A2 There are $0 < M' < M$ such that $M' \leq \dfrac{\partial^2}{\partial\alpha\partial x} u_\alpha(x) \leq M$ for all α, x.

A3 $u_\alpha(\alpha) > 0$ for all $\alpha \in [-1,1]$.

A4 For a fixed α_0, $u_{\alpha_0}(x)$ is strictly concave in x with peak $x = \alpha_0$.

A5 For a fixed x_0, $\frac{\partial u_\alpha(x_0)}{\partial \alpha} > 0$ if $\alpha < x_0$ and $\frac{\partial u_\alpha(x_0)}{\partial \alpha} < 0$ if $\alpha > x_0$.

The essence of assumptions A2-A5 is that agent α has bliss point α and wants to be in the club if the policy x_t is close enough to α; higher agents prefer higher policies; and the set of agents desiring membership is always an interval. A useful example for building intuition is the quadratic case: $u_\alpha(x) = C - (\alpha - x)^2$, where $C > 0$. Finally, we impose the following tie-breaking rule.

A6 An agent α's preferences at time t_0 are as defined by U_α when comparing any two paths $((x_t)_t, I_\alpha)$, $((\tilde{x}_t)_t, \tilde{I}_\alpha)$ with membership rules I_α, $\tilde{I}_\alpha$ that are not both zero for all $t \geq t_0$. However, if $I_{\alpha t} = \tilde{I}_{\alpha t} = 0$ for all $t \geq t_0$, then α prefers $((x_t)_t, I_\alpha)$ to $((\tilde{x}_t)_t, \tilde{I}_\alpha)$ iff $u_\alpha(x_{t_0}) \geq u_\alpha(\tilde{x}_{t_0})$.

In other words, if an agent expects to permanently quit the organization immediately after the current voting stage, she breaks ties in favor of the path with the better current policy. This assumption prevents members who intend to quit from making arbitrary choices out of indifference.[2]

Solution Concept

We will use Markov Voting Equilibrium (MVE) (Roberts 2015; Acemoglu et al. 2015) as our solution concept. This amounts to imposing two simplifying assumptions on our equilibrium analysis. First, rather than explicitly modeling the voting process,

[2] This tie-breaking rule would be uniquely selected if the game were modified to add a small time gap between the voting and membership stages, so that outgoing members at time t would receive a residual payoff $\varepsilon u_\alpha(x_{t_0})$ from the policy x_{t_0} chosen right before they leave.

we assume that only Condorcet-winning policies can be chosen on the equilibrium path, as otherwise a majority could deviate to a different policy. Second, we focus on Markov strategies. That is, when votes are cast at time t, voters only condition on the set of incumbent members, I_{t-1}; when entry and exit decisions are made, the only state variable is the chosen policy, x_t.[3]

Definition 1. Let $\mathcal{L}([-1,1])$ be the Lebesgue σ-algebra on $[-1,1]$. A Markov strategy profile $(\tilde{s}, I)$ is given by a membership function $I : [-1,1] \to \mathcal{L}([-1,1])$, and a policy function $\tilde{s} : \mathcal{L}([-1,1]) \to [-1,1]$ such that $\tilde{s}(I) = \tilde{s}(I')$ whenever I and I' differ by a set of measure zero.

We denote by $s = \tilde{s} \circ I$ the *successor function*. A policy x induces a set of members $I(x)$, who will vote for a policy $\tilde{s}(I(x)) = s(x)$ in the next period. Hence, an initial policy y leads to a policy path $S(y) = (y, s(y), s^2(y), \ldots)$.

Definition 2. An MVE is a Markov strategy profile $(\tilde{s}, I)$ such that:

1. Given a policy x, $\alpha \in I(x)$ iff $u_\alpha(x) \geq 0$.
2. Given a set of voters I, the policy path $S(\tilde{s}(I))$ is a Condorcet winner among the available policy paths. That is, for each $y \neq \tilde{s}(I)$, a weak majority of I weakly prefers $S(\tilde{s}(I))$ to $S(y)$.[4]

From here on we describe equilibria in terms of I and s rather than I and $\tilde{s}$. This is without loss of detail, as the set of voters is always of the form $I(x)$ on the equilibrium path.[5]

We now provide some definitions that will be useful for our analysis. $x \in [-1,1]$ is a *steady state* of a successor function s if $s(x) = x$. x is *stable* if there is a neighborhood $(a,b) \ni x$ such that, for all $y \in (a,b)$, $s^t(y) \xrightarrow[t\to\infty]{} x$. We refer to the largest such neighborhood as the *basin of attraction* of x.

We define the *median voter function* m as follows: for each policy x, $m(x)$ is the median member of the induced voter set $I(x)$, i.e., $\int_{-1}^{m(x)} f(y)\mathbb{1}_{\{\alpha \in I(x)\}}dy = \int_{m(x)}^{1} f(y)\mathbb{1}_{\{\alpha \in I(x)\}}dy$.[6] Finally, we will often be interested in whether an equilibrium

[3]Our solution concept is equivalent to pure-strategy Markov Perfect Equilibrium (MPE), if in each voting stage two office-motivated politicians engage in Downsian competition.

[4]Note that only one-shot deviations are considered: after a deviation to $y \neq s(x)$, it is expected that the MVE will be followed otherwise, i.e., the policy path will be $(y, s(y), \ldots)$ instead of $(s(x), s^2(x), \ldots)$. This is without loss of generality.

[5]If an agent deviates from her equilibrium membership decision, the resulting set of members will differ from $I(x)$ by a set of measure zero, so tomorrow's policy will be unchanged.

[6]$m(x)$ is uniquely defined if $I(x)$ is an interval, as will turn out to be the case.

satisfies the Median Voter Theorem (MVT), i.e., whether the Condorcet-winning policy for a voter set $I(x)$ is also $m(x)$'s optimal choice. Formally, given a successor function s and a set $X \subseteq [-1, 1]$, we will say the MVT holds in X if, for each $x \in X$ and all $y \in [-1, 1]$, $m(x)$ weakly prefers $S(s(x))$ to $S(y)$.

Examples

As an illustration, we map the model to two concrete examples. The first one is Tiebout-style policy competition between cities. Assume that there is a universe of "normal" cities $c \in [-1, 1]$, and a "special" city c^*. Cities differ in two ways. First, each city has a *policy* $x_t(c) \in [-1, 1]$, denoting a certain level of taxation and public goods in city c at time t. For example, a higher x represents higher local taxes which finance better public schools and amenities. Second, c^* has an intrinsic attribute that makes it more desirable than normal cities (good weather, a strong economy, etc.). For simplicity, suppose that each normal city has a positive mass of immobile voters tied to it, and the median immobile voter in city c has bliss point c, so that $x_t(c) = c$ for all t, c. In addition, there is a unit mass of mobile agents in the model, whose bliss points are distributed according to a density f. $x_0(c^*)$ is exogenous.

We are interested in the policy path of c^* and the behavior of mobile agents. At each time t, each mobile agent α chooses a city to live in. Her flow payoff from choosing c is $u_\alpha(x, c) = C\mathbb{1}_{c=c^*} - (x_t(c) - \alpha)^2$, where $C > 0$ is the intrinsic value of c^*. Clearly her decision boils down to a binary choice: she should live either in c^* or in her most-preferred normal city, $c = \alpha$, yielding a flow payoff of zero. Living anywhere but in c^* is equivalent to leaving the club in the general model.

The second example we discuss is that of trade unions. Assume an economy with a unionized firm and a larger competitive (non-union) sector. Firms offer employment contracts (w, l) consisting of a wage w and a family leave policy l. The marginal productivity of all workers is normalized to 1, and a leave policy l signifies that the worker only works a fraction $1 - l$ of the time. In equilibrium, competitive firms are willing to offer any contract of the form $(1-l, l)$; the competitive sector is assumed to be large enough that all such contracts are available. The union, through collective bargaining, extracts a wage $w_u > 1$ from the unionized firm, so its leadership can bargain for any contract of the form $(w_u - l, l)$, but the same contract must apply to all unionized workers. As in Grossman (1983), assume the union bargains on behalf of its median voter.

Workers differ in their taste for family leave. A worker of type α has flow payoff $\tilde{u}_\alpha(w,l) = w + \alpha v(l)$, where $v(0) = 0$ and v is smooth, increasing and strictly concave in l. Workers can move freely between firms, including to the unionized firm; upon joining the latter, they automatically become union members.[7]

Of all the competitive firms, a worker α prefers to join one offering $l = l^*(\alpha)$, where $v'(l^*(\alpha)) = \frac{1}{\alpha}$. Let $u_\alpha(l) = \tilde{u}_\alpha(w_u - l, l) - \tilde{u}_\alpha(1 - l^*(\alpha), l^*(\alpha))$ be α's net utility from joining the union sector when the union has bargained for a contract $(w_u - l, l)$. Up to a relabeling, u satisfies A1-5 and hence the model applies without changes.

3 Equilibrium Analysis

In this Section we prove some fundamental properties of all MVEs, which in particular allow us to characterize the club's policy in the long run. We start by solving for the equilibrium membership strategy, which is simple:

Lemma 1. *In any MVE, $I(y) = [y - d_y^-, y + d_y^+]$ is an interval, and $d_y^-, d_y^+ > 0$ are given by the condition $u_{y-d_y^-}(y) = u_{y+d_y^+}(y) = 0$.*

Since members can enter or leave at any time, it is optimal for α to join whenever the flow payoff of the current policy, $u_\alpha(x)$, is positive, and leave when it is negative; the Lemma then follows from Assumptions A3 and A5. An immediate corollary is that m is strictly increasing and C^1. Additionally, since $I(x)$ is uniquely determined, we can describe MVEs solely in terms of successor functions.

Before characterizing s in general, it is instructive to consider two simple special cases. First, suppose that $I(y) = I$ is independent of y (for instance, $I(y) \equiv [-1, 1]$, i.e., everyone always prefers to be in the club). In this case, regardless of the current policy y, the Condorcet winner is the bliss point of the median member of I. Second, suppose that $\delta = 0$, i.e., agents are myopic. Given an initial policy y and set of members $I(y)$, the Condorcet winner is the bliss point of $m(y)$, and the policy path will be $(y, m(y), m^2(y), \ldots)$, which converges to a myopically stable policy $m^*(y) = \lim_{k\to\infty} m^k(y)$. In both scenarios, the simplicity of the solution stems from the lack of tension between current payoffs and future control: in the former case there is no link between them, while in the latter case they are linked but voters do not care.

[7] This is a common arrangement is known as a "union shop".

As a first step towards solving the general case we show that equilibrium paths are always monotonic.

Proposition 1. *In any MVE, for any y, if $s(y) \geq y$ then $s^k(y) \geq s^{k-1}(y)$ for all k, and if $s(y) \leq y$ then $s^k(y) \leq s^{k-1}(y)$ for all k.*

To see why this must be the case, imagine an equilibrium path $(x_0, x_1, \ldots)$ that increases up to x_k ($k > 0$) and decreases afterwards. Then $S(x_k)$ must be a Condorcet winner in $I(x_{k-1})$, and $S(x_{k+1})$ must be a Condorcet winner in $I(x_k)$. In particular, a majority in $I(x_{k-1})$ must prefer $S(x_k)$ to $S(x_{k+1})$ but a majority in $I(x_k)$ must prefer the opposite. This is impossible because $S(x_{k+1})$ has a lower average policy than $S(x_k)$, while the group $I(x_k)$ contains agents with higher bliss points than $I(x_{k-1})$.[8] Our next result characterizes the long-run behavior of any MVE satisfying the MVT.

Proposition 2. *In any MVE s and for any y:*

(i) If $m(y) = y$ then $s(y) = y$.

(ii) If $m(y) > y$ then $m^*(y) > s(y) \geq y$. Moreover, if the MVT holds in $[y, m^*(y)]$ then $s(y) > y$ and $s^k(y) \xrightarrow[k\to\infty]{} m^*(y)$.

(iii) If $m(y) < y$ then $m^*(y) < s(y) \leq y$. Moreover, if the MVT holds in $[m^*(y), y]$ then $s(y) < y$ and $s^k(y) \xrightarrow[k\to\infty]{} m^*(y)$.

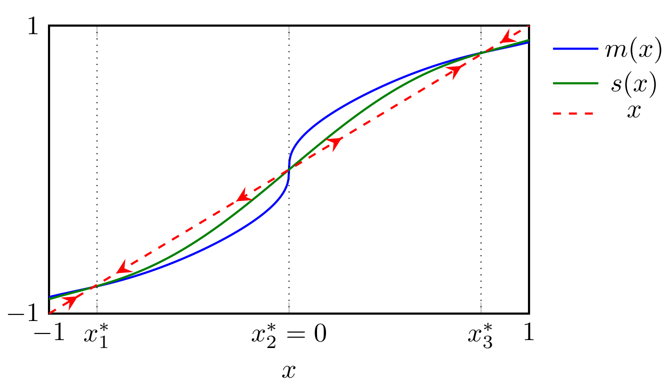

Figure 1: Convergence to steady states in MVE

[8]Analogous results are shown in Roberts (2015) and Acemoglu et al. (2015). The proof here is more involved because, owing to the infinite policy space, we have to rule out non-monotonic paths that never reach their supremum or infimum.

In other words, the steady states of any MVE s satisfying the MVT are simply the fixed points of the mapping $y \mapsto m(y)$. Moreover, stable (unstable) steady states of s are also stable (unstable) fixed points of m, and their basins of attraction coincide.

The intuition for why we should observe $s(x) \leq x$ if $m(x) < x$ and vice versa is straightforward: if $m(x) < x$ for x in an interval (x^*, x^{**}), any policy in that interval attracts a set of voters whose median wants a lower policy. The main conclusion of Proposition 2 is that slippery slope concerns cannot create myopically unstable steady states, that is, $s(x) \neq x$ if there is a myopic incentive to change the policy. The logic behind the proof is as follows. Suppose $m(x) < x$, but $m(x)$ is afraid of further policy changes if she moves to any $y < x$. If $m(x)$ chooses a *slightly* better policy $y = x - \epsilon$, her flow payoff tomorrow will increase by roughly $\epsilon \left|\frac{\partial u}{\partial x}\right|$. In exchange, she will relinquish control over the continuation to a slightly different voter, $m(x-\epsilon)$. Because they have similar preferences (Assumption A2), $m(x-\epsilon)$'s optimal choice is also approximately optimal for $m(x)$. Hence the cost of losing control is small, that is, no higher than $M(m(x) - m(x-\epsilon)) \sum_t \delta^k |s^k(x) - s^k(s(x-\epsilon))|$. If $S(s(x-\epsilon))$ converges to $S(x)$ pointwise as $\epsilon \to 0$, this loss is of order $o(\epsilon)$, so $m(x)$ should deviate to $y = x - \epsilon$ for ϵ small enough. If not, it can be shown that $m(x)$ must be indifferent between $S(x)$ and $\lim_{\epsilon \to 0} S(s(x-\epsilon))$, and an analogous argument can then be made.

Figure 1 illustrates the equilibrium properties stated in Proposition 2 in an example with three steady states: x_1^* and x_3^* are stable, while x_2^* is unstable. This alternation of stable and unstable steady states occurs in general as long as m is well-behaved. Formally, in the rest of the paper we will assume the following:

B1 The equation $m(y) = y$ has finitely many solutions $x_1^* < x_2^* < \ldots < x_N^*$. In addition, $m'(x_i^*) \neq 1$ for all i.

Corollary 1. *m has an odd number of fixed points. For odd i, $m'(x_i^*) < 1$ and x_i^* is a stable steady state of every MVE; for even i, $m'(x_i^*) > 1$ and x_i^* is unstable.*

Our last result in this Section guarantees that, in a sizable neighborhood of each stable steady state, every equilibrium satisfies the MVT, and hence the main conclusion of Proposition 2 applies. In addition, within the same neighborhood every equilibrium must be monotonic (a stronger property than path-monotonicity) and an equilibrium restricted to this neighborhood must exist.

Proposition 3. *Let x^* be such that $m(x^*) = x^*$ and $m'(x^*) < 1$; let $x^{***} < x^* < x^{**}$ be the unstable steady states adjacent to x^*. Then an MVE restricted to $I(x^*) \cap (x^{***}, x^{**})$ exists,*[9] *and any MVE is weakly increasing and satisfies the MVT in $I(x^*) \cap (x^{***}, x^{**})$.*

The reason these results may fail to hold outside of $I(x^*) \cap (x^{***}, x^{**})$ is that they rely on pivotal voters not leaving the club on the equilibrium path; when pivotal voters quit the club at different times, the logic that voters with higher bliss points should like higher paths does not always apply.[10]

We finish this Section with two remarks. First, an alternative approach to solving for MVEs would be to study a game in which, given a policy x, the agent $m(x)$ is by assumption given direct control over tomorrow's policy.[11] In this closely related game, the main conclusion of Proposition 2 holds for all Markov equilibria. Second, as we will see next, under some conditions we will be able to guarantee the existence of MVE that are monotonic and satisfy the main conclusion of Proposition 2 everywhere.

4 Transition Dynamics

This Section analyzes the transition dynamics of the model in more detail. Without loss of generality, we restrict our analysis to the right side of the basin of attraction of a stable steady state, that is, an interval $[x^*, x^{**})$ such that $m(x^*) = x^*$, $m(x^{**}) = x^{**}$ and $m(y) < y$ for all $y \in (x^*, x^{**})$.[12]

We begin by noting that, under mild conditions, convergence to a steady state is far from instant, and becomes arbitrarily slow if agents are arbitrarily patient. Formally, say $m(x) \in (x^*, x^{**})$ is *reluctant* if $u_{m(x)}(x) > u_{m(x)}(x^*)$, i.e., $m(x)$ would rather stay at x than move instantly to x^*.[13] If so, let $z(x)$ be the unique policy in $(x^*, m(x))$ for which $u_{m(x)}(x) = u_{m(x)}(z(x))$.

Proposition 4. *If $m(x)$ is reluctant, then, for all $y < z(x)$, $\exists K(y) > 0$ such that, for any δ and any MVE s of the game with discount factor δ, $\min\{t : s^t(x) \leq y\} \geq \frac{K(y)\delta}{1-\delta}$.*

[9] s is a MVE restricted to $[a, b]$ if it is a MVE when the policy space is restricted to $[a, b]$.

[10] For instance, let $\alpha = 0.6$, $\tilde{\alpha} = 0.5$, and $S = (0.7, 0.1)$, $T = (0.65, 0)$ be two-period policy paths. If α, $\tilde{\alpha}$ never leave under either path, $U_\alpha(S) - U_\alpha(T) > U_{\tilde{\alpha}}(S) - U_{\tilde{\alpha}}(T)$ by A2; but if $u_\alpha(0) < 0$ it is possible that $U_\alpha(S) - U_\alpha(T) < U_{\tilde{\alpha}}(S) - U_{\tilde{\alpha}}(T)$. By the same logic, the set of voters preferring S to T may not be an interval, so a winning coalition need not contain the median voter.

[11] This approach has been taken in the literature, e.g., in Bai and Lagunoff (2011).

[12] For any $y \in (x^*, x^{**})$, $I(y)$ never wants to choose a policy outside of (x^*, x^{**}), so $s|_{(x^*, x^{**})}$ can be studied in isolation. Results for a basin of attraction of the form $[x^*, 1]$ or $[x^{***}, x^*]$ are analogous.

[13] For instance, in the quadratic case, if $m'(x) > \frac{1}{2}$ for all x then every agent is reluctant.

The reason is simply that, if this condition were violated, $m(x)$ would rather stay at x forever.[14] Thus, if there are reluctant agents, knowing the club's long-run policy is not enough to characterize the agents' utility even as $\delta \to 1$, unlike in related models (cf. Acemoglu et al. 2012); further analysis of the transition path is necessary.

In the remainder of this Section we propose a natural class of equilibria, which we call *1-equilibria* (henceforth 1Es), and study their transition dynamics. There are two reasons to focus on 1Es. The first is tractability: 1Es have a simple structure, and their transition dynamics can be explicitly characterized in the limit as agents become more patient. The second is robustness: 1Es are guaranteed to exist under some conditions we will provide, while other types of equilibria (including smooth equilibria) cannot be guaranteed to exist. We begin with a definition of 1Es and two related concepts.

Definition 3. Let s be a successor function on $[x^*, x^{**}]$. s is a *1-function* if there is a sequence $(x_n)_{n\in\mathbb{Z}}$ such that $x_{n+1} < x_n$ for all n, $x_n \xrightarrow[n\to-\infty]{} x^{**}$, $x_n \xrightarrow[n\to\infty]{} x^*$, and $s(x) = x_{n+1}$ if $x \in [x_n, x_{n-1})$. We call $(x_n)_n$ the *recognized sequence* of s.
s is a *1-equilibrium* (1E) if it is a 1-function and an MVE.
s is a *quasi-1-equilibrium* (Q1E) if it is a 1-function such that $(1-\delta)U_{m(x_n)}(S(x_{n+1})) = u_{m(x_n)}(x_{n+1})$ for all n.

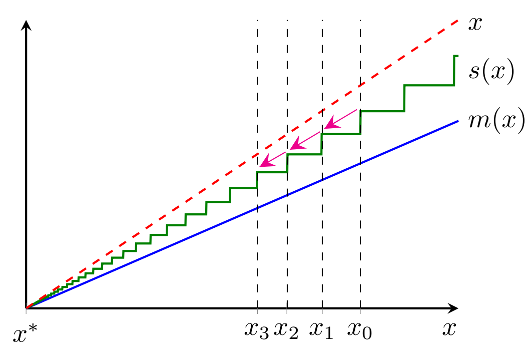

Figure 2: 1-equilibrium for $u_\alpha(x) = C - (\alpha - x)^2$, $m(x) = 0.7x$, $\delta = 0.7$

In a 1E, the chosen policies are always elements of the recognized sequence $(x_n)_n$. x_n today leads to x_{n+1} tomorrow; if the initial policy is not part of the recognized sequence, but is between x_n and x_{n-1}, then x_{n+1} is chosen, and the path follows the

[14]A partial converse holds: if $u_{m(x)}(x^*) > u_{m(x)}(x)$ for all x then, for all $y \in (x^*, x)$, $\min\{t : s^t(x) \le y\} \le \tilde{K}(y)$, with $\tilde{K}(y)$ independent of δ.

recognized sequence thereafter. An illustration is given in Figure 2. The notion of Q1E is useful to study because Q1Es are closely related to 1Es, but are easier to construct. The following Proposition summarizes the relationship between the two.

Proposition 5.

(i) Every 1E is also a Q1E.

(ii) A Q1E exists. Moreover, for each $\underline{x} \in (x^*, x^{**})$, there is a Q1E $s_{\underline{x}}$ with $x_0 = \underline{x}$.

(iii) In any Q1E, for all $x \in [x_n, x_{n-1})$ and for all k, a majority in $I(x)$ prefers $S(x_{n+1})$ to $S(x_k)$.

(iv) Any Q1E such that $m(x_n) < x_{n+2}$ for all n is a 1E within $[x^*, m^{-1}(x^* + d^+_{x^*})]$.

To see why 1Es are also Q1Es, consider a 1E s with recognized sequence $(x_n)_n$. By construction, a majority in $I(x_n)$ prefers $S(x_{n+1})$ to $S(x_{n+2})$. But, for any x in a left-neighborhood of x_n, a majority of $I(x)$ prefers $S(x_{n+2})$ to $S(x_{n+1})$. Due to the fact that $S(x_{n+1}) = (x_{n+1}, S(x_{n+2}))$, a voter α prefers $S(x_{n+1})$ to $S(x_{n+2})$ iff $u_\alpha(x_{n+1}) \geq (1-\delta)U_\alpha(S(x_{n+1}))$, and it can be shown that the set of voters with this preference is of the form $[\alpha^*, 1]$. Then we must have $\alpha^* = m(x_n)$. By continuity, $m(x_n)$ is indifferent between $S(x_{n+1})$ and $S(x_{n+2})$, i.e., $(1-\delta)U_{m(x_n)}(S(x_{n+1})) = u_{m(x_n)}(x_{n+1})$.

Part (ii) of Proposition 5 guarantees that Q1Es always exist—in fact, there is a continuum of them. Part (iii) is a partial converse of (i): it shows that, in a Q1E, no deviations to other policies on the recognized sequence are possible. To see why $I(x_{n-1})$ will not deviate to $S(x_{n+2})$, for instance, note that $m(x_n)$ is indifferent between $S(x_{n+1})$ and $S(x_{n+2})$, so $m(x_{n-1})$ prefers $S(x_{n+1})$ to $S(x_{n+2})$, and is indifferent between $S(x_n)$ and $S(x_{n+1})$ by construction. Equivalently, then, the only reason a Q1E may fail to be a 1E is if a majority wants to deviate off the recognized sequence.

Intuitively, such deviations will not be desirable if the median voter, $m(x_n)$, is far to the left of the first few elements of the sequence following x_n. To see why, note that, if $m(x_n) \in [x_k, x_{k-1})$, deviating to a policy $y \in (x_l, x_{l-1})$ with $l < k$ would be even worse than deviating to x_l, while deviating to $y \in (x_l, x_{l-1})$ with $l > k$ would be worse than deviating to x_{l-1}. Hence the only deviations $m(x_n)$ might prefer are deviations to $y \in (x_k, x_{k-1})$. In particular, picking $y = m(x_n)$ is better than deviating to x_k. But such a deviation will still be unprofitable if $S(x_{n+1})$ is too strongly preferred to

$S(x_k)$. This is the idea behind part (iv); a more powerful result along these lines will be given in the next subsection.

Finally, note that, in a 1E, $m(x_n)$'s averaged per-period utility equals $u_{m(x_n)}(x_{n+1})$. Hence her net gain from not staying at x_n forever, $V(m(x_n)) := (1-\delta)U_{m(x_n)}(S(x_{n+1})) - u_{m(x_n)}(x_n)$, is approximately proportional to the equilibrium speed of policy change; specifically, it equals $(x_n - x_{n+1})\left|\frac{\partial u_{m(x_n)}(y)}{\partial y}\right|$ for some $y \in [x_{n+1}, x_n]$.

Continuous Time Limit

We now characterize the limit of 1Es as the time gap between rounds of voting becomes arbitrarily small. This can be taken as an approximation of a setting in which voting happens periodically (e.g., at annual elections), but often enough relative to the agents' time horizon. The same results will also allow us to characterize the limit of 1Es with a fixed time gap between periods as $\delta \to 1$.

Denote $\delta = e^{-r}$. We will work with the following objects. First, for each $j \in \mathbb{N}$, consider a version of the game from Section 2 in which policy and membership decisions are made at every time $t \in \{0, \frac{1}{j}, \frac{2}{j}, \ldots\}$ instead of at every integer time. We will call this the j-refined game, and denote a Q1E of this game by s_j. In addition, for each $t \in \mathbb{R}_{\geq 0}$, we denote by $s_j(x,t)$ the equilibrium policy at time t if the initial policy is x and s_j is played—that is, $s_j(x,t) = s_j^{\lfloor tj \rfloor}(x)$. Note that this game is, up to a relabeling, equivalent to the model in Section 2 with discount factor $\delta_j = e^{-\frac{r}{j}}$.

Finally, we define a continuous limit solution (CLS) as a function $s(x,t) : [0,+\infty) \times [x^*, x^{**}) \to [x^*, x^{**})$ with the following properties: $s(x, t+t') \equiv s(s(x,t'),t)$; $s(x,0) \equiv x$; s is weakly decreasing in t; $s(x,t) \xrightarrow[t \to 0]{} x$ for all x; and $\overline{U}_{m(x)}(S(x)) = u_{m(x)}(x)$ for all $x \in (x^*, x^{**})$, where $\overline{U}_\alpha(S(x)) = \int_0^\infty re^{-rt} \max(u_\alpha(s(x,t)), 0) dt$.

The following Proposition relates the CLS to the Q1Es of the j-refined games.

Proposition 6. *Suppose that $m \in C^2$ and a CLS s exists. Then:*

(i) s is the unique CLS, and s is C^1 as a function of t.

(ii) For any sequence $(s_j)_j$, where s_j is a Q1E of the j-refined game, for all x and t, $s_j(x,t) \xrightarrow[j \to \infty]{} s(x,t)$.

(iii) There is $\overline{\delta} < 1$ such that, for all $\delta > \overline{\delta}$, all Q1Es of the discrete-time game with discount factor δ are 1Es.

The intuition behind a CLS is the following. Fix x, and take a sequence $(s_j)_j$ of Q1Es of the j-refined games with $x_{j0} = x$. Recall that $U_{m(x_{jn})}(S_j(x_{j(n+1)})) = u_{m(x_{jn})}(x_{j(n+1)})$ for all j, n.[15] Suppose that, as $j \to \infty$, the transition paths $S_j(x) = (s_j(x,t))_t$ converge pointwise to a continuous path $S(x)$. Then the differences $x_{j0} - x_{j1}$ go to zero, and in the limit, $U_{m(x)}(S(x)) = u_{m(x)}(x)$. This is why we require this condition of a CLS. Part (ii) of Proposition 6 is a converse to this argument: it shows that, when a CLS exists, the transition paths of all Q1Es must converge to it as $j \to \infty$.

Whether a CLS exists is a property of the primitives u and m; it can be determined in isolation from our game. We can both verify whether a CLS exists and calculate it explicitly, as follows. Denote by $e(x) = -\frac{1}{\frac{\partial s(x,t)}{\partial t}|_{t=0}}$ the instantaneous delay of a CLS at x. If $U_{m(x)}(S(x)) = u_{m(x)}(x)$ for all x, then, differentiating with respect to x,

$$m'(x)\frac{\partial U_{m(x)}(S(x))}{\partial \alpha} = m'(x)\frac{\partial u_{m(x)}(x)}{\partial \alpha} + \frac{\partial u_{m(x)}(x)}{\partial x}. \tag{1}$$

The key observation is that $\frac{\partial}{\partial x}\frac{\partial U_m(S(x))}{\partial \alpha} = \left(\frac{\partial u_{m(x)}(x)}{\partial \alpha} - \frac{\partial U_m(S(x))}{\partial \alpha}\right) re(x)$. Hence, differentiating Equation 1 yields an equation that pins down $e(x)$. After rearranging, we find

$$-\frac{\partial u}{\partial x} re(x) = 2m'\frac{\partial^2 u}{\partial \alpha \partial x} + \frac{\partial^2 u}{\partial x^2} + (m')^2\left(\frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 U_{m(x)}(S(x))}{\partial \alpha^2}\right) - \frac{m''}{m'}\frac{\partial u}{\partial x}, \tag{2}$$

where u stands for $u_{m(x)}(x)$. This is an integral equation because $\frac{\partial^2 U}{\partial \alpha^2}$ is evaluated at $S(x)$, which depends on $e(y)$ for $y \in [x^*, x)$. But we can solve it forward, starting at x^*, to find $e(x)$ and, with it, the unique CLS. In fact, a CLS always exists unless the function $e(x)$ that solves Equation 2 turns negative. This is guaranteed not to happen if the forces pulling towards policy change are not too great:

Proposition 7. *Holding u constant, there exist B, B', $B'' > 0$ such that, if $x - m(x) \leq B$, $m'(x) \in [1 - B', 1 + B']$ and $m''(x) \geq -B''$ for all x, then a CLS exists.*

The quadratic case is useful for illustrative purposes. In that case, for $x \in I(x^*)$, Equation 2 reduces to

$$re(x) = \frac{2m'(x) - 1}{x - m(x)} + \frac{m''(x)}{m'(x)}, \tag{3}$$

[15] In this argument, we write $U_{m(x)}((x_t)_t) = (1-\delta)\sum \delta^t I_{\alpha t} u_\alpha(x_t)$ to simplify notation.

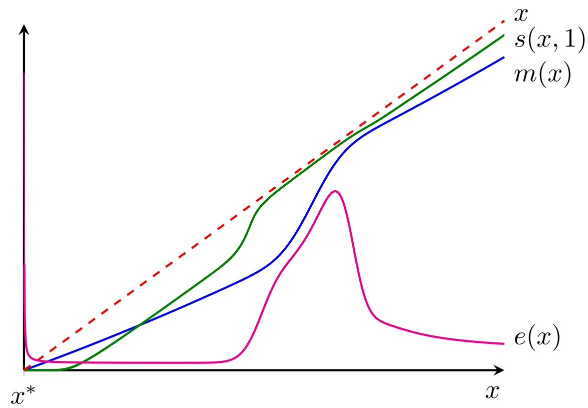

Figure 3: A continuous limit solution

so the speed of convergence can be calculated in closed form. Equation 3 reflects three forces at work in determining $e(x)$. First, the policy changes more slowly ($e(x)$ is higher) when the myopic incentive for policy changes, $x - m(x)$, is small. Second, the policy changes more slowly when $m'(x)$ is high. The reason is that changing the policy to $x-\epsilon$ entails yielding control to another agent $m(x-\epsilon)$. The higher m' is, the more costly this loss of control becomes. Third, the policy changes more slowly when $m''(x)$ is high. The reason is that, when m is convex, $m'(x)$ is higher than $m'(x-\epsilon)$; hence the agent $m(x)$ yields control to will not be as concerned about the behavior of her own successors, and so will make faster policy changes than $m(x)$ would like. These forces are illustrated in Figure 3.

Finally, part (iii) of Proposition 6 guarantees that, when there is a CLS, all Q1Es must be true 1Es when agents are patient enough. The reason is related to our discussion of Proposition 5: as the transition paths of Q1Es approach the CLS, they must feature smaller and smaller jumps between consecutive policies; in this scenario, deviations off the recognized sequence are never majority-preferred.

We conclude this Section with a few observations. First, the speed of policy change in a CLS is exactly inversely proportional to the agents' patience:

Remark 1. If $s(x,t)$ is a CLS for discount rate r, then $\tilde{s}(x,t) \equiv s\left(x, \frac{\tilde{r}}{r}t\right)$ is a CLS for discount rate $\tilde{r}$. The respective instantaneous delays $e(x)$, $\tilde{e}(x)$ satisfy $\tilde{e}(x) = \frac{r}{\tilde{r}}e(x)$.

The reason is that changing r is equivalent to a relabeling of the time variable. Second, when a CLS exists, Proposition 6 and Equation 2 together yield an asymptotic characterization of the transition path for all 1Es of the game from Section 2 when agents are patient.[16] Formally, let $e_1(x)$ be the solution to Equation 2 for $r = 1$. Then, for any $y < x$ and any collection of 1Es s_δ for $\delta \geq \overline{\delta}$,

$$(1-\delta)\min\{t : s_\delta^t(x) \leq y\} \underset{\delta \to 1}{\longrightarrow} \int_y^x e_1(z)dz. \tag{4}$$

This is, in effect, a more precise version of Proposition 4. Third, recall that, in a 1E, the net per-period gain $V(m(x_n))$ of a pivotal agent $m(x_n)$ from following the equilibrium path (relative to staying at x_n) is of the order $x_n - x_{n+1}$. Thus, if a CLS exists, for any x and any collection of 1Es s_δ, $V_\delta(m(x)) \to 0$ as $\delta \to 1$. In other words, the "rents" a pivotal agent gets from the best non-constant continuation evaporate as agents become more patient (or decisions are made more often). An intuition is that these rents are the result of agents being able to "lock in" their chosen policy for one period before losing control—hence they vanish as the periods shorten.

Fourth, it is not hard to find examples in which a CLS fails to exist.[17] However, even when there is no CLS, a version of Proposition 6 holds: under some conditions, Q1Es can still be guaranteed to be 1Es for high δ, and all sequences of Q1E transition paths converge to a common limit, but this limit is no longer continuous. The details for this case are presented in Appendix B.

5 A Model of Political Power

We now discuss a variant of the model that overturns the assumption of free entry and exit. Consider a polity governed by an endogenous *ruling coalition*. At each time $t = 1, 2, \ldots$ the ruling coalition chooses a policy x_t; the initial policy x_0 is exogenous.

The model is the same as the one presented in Section 2, but with two differences. First, the policy x_t now directly determines not just the flow payoffs of all agents during the period $[t, t+1)$, but also the ruling coalition at time $t+1$. In other words, the mapping $x \mapsto I(x)$ is now taken as a primitive of the model. (We assume that $I(x)$ is still an interval $(x - d_x^-, x + d_x^+)$ for each x, with $x - d_x^-$, $x + d_x^+$ increasing and C^1 as functions of x.) Second, in this model, *all* agents are impacted by the policy,

[16] This is a consequence of Proposition 6 for a sequence $(\delta_j)_j$ of the form $\delta_j = e^{-\frac{r}{j}}$, but in fact the proof of the Proposition does not rely on the j's being integers, only that $j \to \infty$.

[17] For example, if there is a non-reluctant agent, then a CLS cannot exist.

regardless of whether they are in the ruling coalition. In other words,

$$U_\alpha\left((x_t)_t, I_\alpha\right) = \sum_{t=0}^{\infty} \delta^t u_\alpha(x_t),$$

where u_α satisfies A1, A2 and A4. This setting is similar to the canonical model of "elite clubs" (Roberts, 2015), but with a continuous policy space. It can be framed as a model of enfranchisement (Jack and Lagunoff, 2006), institutional change (Acemoglu et al., 2012, 2015), or economic policymaking in a world where political influence is a function of wealth (Bai and Lagunoff, 2011). Clearly, slippery slope concerns apply here as well: a ruling coalition may want to expand the franchise (e.g., to lower unrest) but fear that the new voters will choose to expand it even further.

Our analysis of the main model extends to this case as follows. First, all of our previous Propositions continue to hold, with analogous proofs. Second, the conclusions of Propositions 3 and 5(iv) now hold for all $x \in [-1, 1]$, as opposed to only in a neighborhood of each stable steady state.[18] In particular, an MVE exists; every MVE satisfies the MVT for all x; and for every MVE $s^k(x) \xrightarrow[k\to\infty]{} m^*(x)$ for all x.

Although this version of the model represents a setting with different causal relationships between political power, membership, and flow payoffs, it is closely related to the model from Section 2: indeed, there is a mechanical equivalence between the components of both models, and the same tension between current payoffs and future control is present in both cases.

Other variants, allowing the model to fit new examples, are possible. For instance, the set of members can affect payoffs directly: $v_\alpha(x) = u_\alpha(x) + w_\alpha(I(x))$. So long as $v_\alpha(x)$ satisfies A1-4, all of our results apply. A natural example is immigration: if x_t is a country's immigration policy and $I(x_t)$ is its set of citizens, x_t does not affect the payoffs of current citizens directly, but the entry of immigrants does.[19]

6 Discussion

This Section discusses some implications of our analysis.

[18]The proofs of Propositions 3 and 5(iv) assume that pivotal agents never stop receiving payoffs from the club's policy. In the main model, this requires them not to leave the club; in this variant it does not matter if they are part of the ruling coalition.

[19]This example has been studied in the literature, although in an overlapping-generations framework (Ortega, 2005; Suwankiri, Razin and Sadka, 2016).

Myopic Stability of Steady States

Two central results of our analysis are that steady states are myopically stable (or, in the language of Roberts (2015), "extrinsic"),[20] and convergence to a steady state is slow when agents are patient. In contrast, in other papers in this literature (Roberts, 2015; Acemoglu et al., 2008, 2012, 2015), intrinsic steady states are possible, and the time it takes to converge to a steady state is uniformly bounded even as $\delta \to 1$.

These papers assume a fixed, finite policy space. Under this assumption, convergence is fast because there is a mechanical lower bound on the size of policy changes; as a result, intrinsic steady states *must* exist for δ high enough, if there are reluctant agents. What we show is that these results are overturned if arbitrarily small policy changes are allowed. For a fixed $\delta < 1$, our results also hold if the policy space is finite but fine enough; if we simultaneously take δ to 1 and make the policy space arbitrarily fine, whether intrinsic steady states exist depends on the order of limits.

The upshot is that, in practice, whether dynamic concerns can indefinitely stall policy changes depends not just on the agents' foresight but also on institutional details that determine whether incremental changes are possible. For example, take a polity with a limited franchise considering a franchise extension on the basis of income. Suppose that each voter prefers a larger franchise than the smallest one she would be in (e.g., for all x, a voter at the xth income percentile wants to enfranchise everyone above the $(x-5)$th percentile). Then, if it is possible to enfranchise the top $y\%$ of the income distribution for any y, full democracy would eventually be reached through a series of small changes. However, if voting rights can only be extended based on a few criteria (e.g., only to men who can read, to landowners, to taxpayers, etc.), indefinite stalling is likely.[21]

Two important precursors to our analysis, Jack and Lagunoff (2006) and Bai and Lagunoff (2011), consider dynamic political decision-making with continuous policy spaces. In particular, Bai and Lagunoff (2011) show that in their model, steady states of "smooth" equilibria are also stable when the current decision-maker is assumed to retain power forever (in our setting, this is equivalent to myopic stability). However, their analysis uses a first-order approach, and so does not extend to other types of equilibria; moreover, smooth equilibria generically fail to exist, as the second-order

[20] This is true for all 1Es; for all other MVEs in a neighborhood of each stable steady state; and globally for all MVEs in the model discussed in Section 5, which is closest to this literature.

[21] Jack and Lagunoff (2006) make the case that franchise extensions are typically gradual processes.

conditions are typically violated.[22] We build on this result by showing that steady states must be myopically stable for all equilibria, including discontinuous ones.

Distribution of Steady States

Although f is the primitive of our model describing the distribution of preferences, our results are best stated in terms of m, the median voter function. Here, we briefly discuss the relationship between the two objects, focusing on how the shape of f affects the location of steady states.

In the quadratic case, or more generally whenever $I(x) \equiv (x-d, x+d)$ is symmetric around x, the distribution of steady states reflects the following intuition: if f is increasing within $I(x)$ then $m(x) > x$, and vice versa. Hence, stable steady states correspond roughly to maxima of the density function.

Remark 2. If $I(x) = (x-d, x+d)$ for all x, and x^* is a stable (unstable) steady state, then $I(x^*)$ contains a local maximum (minimum) of f.

In particular, if f is increasing (decreasing) everywhere, there is a unique steady state close to 1 (-1); if f is symmetric and single-peaked, 0 is the unique steady state. Thus, the organization always moves to the center if the distribution of preferences is bell-shaped. Yet, there are three scenarios in which the organization may converge to a policy more extreme than the bliss points of most voters.

First, even if most voters are near the center, a local maximum near an extreme may support a stable steady state, especially if d is low.[23]

Remark 3. If $f'(x^*) = 0$ and $f''(x^*) < 0$, then there is $\overline{d} > 0$ such that for all $d < \overline{d}$, if $I(x) \equiv (x-d, x+d)$ for all x, $(x^* - d, x^* + d)$ contains a stable steady state.

Second, even if there is a unique steady state, its location will be unstable when f is close to uniform. For example, consider the densities $f_1(x) = \frac{1}{2} + \epsilon x$, $f_2(x) = \frac{1}{2} - \epsilon x$ and $f_3(x) = \frac{1+\epsilon}{2} - \epsilon |x|$, for a small $\epsilon > 0$. These are all close to each other ($||f_i - f_j||_\infty \leq 2\epsilon$ $\forall i, j$) but f_1 has a unique steady state near -1, f_2 has one near 1, and f_3 has one at 0. Hence, the long-run policy is potentially discontinuous in f, unlike in models of voting with a fixed population.

[22] The existence properties of smooth equilibria are discussed in Appendix D.

[23] Note that, when there are multiple steady states, which one the organization converges to depends on the initial policy, i.e., there is path-dependence. There is no guarantee that the organization will converge to a steady state that attracts the most members or maximizes aggregate welfare.

Third, the tendency towards policies preferred by a majority can be easily overturned when the voter sets $I(x)$ are not symmetric around x. In particular, if agents with extreme preferences are disproportionately more willing to join the organization, they can capture it despite being a minority, even locally.[24]

For example, let the policy space be $[-1, 1]$, where -1 is the most moderate policy and 1 the most extreme, and assume $u_\alpha(x) = -|\alpha - x| + (1 + \alpha)$.[25] Then α wants to be a member whenever $x \in [-1, 2\alpha + 1]$, whence $I(x) = [\frac{-1+x}{2}, 1]$.

Suppose f is as follows: moderates constitute 60% of the population and have bliss points uniformly distributed in [-1,-0.9]; extremists, the remaining 40%, have bliss points uniformly distributed in [-0.9,1]. It can then be shown that the unique steady state is $x^* = \frac{1}{3} > 0$. At the steady state policy, the set of members $I(\frac{1}{3}) = [-\frac{1}{3}, 1]$ is only 28% of the population, all of them extremists.

7 Conclusion

We conclude with a discussion of some issues that the model leaves out, possible extensions, and additional results presented in the Appendix.

Our model focuses on the behavior of a single organization, but organizations often compete—in particular, the usual assumption in Tiebout competition is that there are many districts. There are two ways of modeling competition. The first is to assume, as in the idealized Tiebout model, that districts are identical except for their policies. In such a model, the same dynamics we have studied would arise, but with the complication that policy changes in one district may lead to responses by other districts. At the same time, if there are enough districts, every agent would find a district near her bliss point, so the potential welfare impact of policy changes in individual districts would vanish as the number of districts grows.[26]

An alternative approach is to assume that districts are imperfect substitutes. The example in Section 2 of a city with a competitive advantage over others is a version of this, and it can be generalized. Suppose that there are $k > 1$ special cities c_1^*,

[24] This asymmetry is likely in settings where too-extreme policies are perceived as reprehensible or criminal, but not the reverse (e.g., fringe political parties, violent protest movements, or advocacy groups whose causes can be perceived as racist or xenophobic).

[25] The example is degenerate in that $\frac{\partial^2 u}{\partial \alpha \partial x}$ is only weakly positive and u is only weakly concave in x; this is only for simplicity.

[26] Tiebout (1956) conjectured that agents sorting into compatible districts would lead to an efficient outcome. This idea has been formalized (Wooders, 1989) as well as criticized (Bewley, 1981).

…, c_k^* and k groups of mobile agents, so that agent types are of the form (α, i), for $i \in \{1, \ldots, k\}$, and group i's bliss points are distributed according to a density f_i. Assume that $u_{(\alpha,i)}(c) = C\mathbb{1}_{c=c_i^*} - (x_t(c) - \alpha)^2$, that is, agents in group i only value the city c_i^* more than normal cities. (For instance, agents in group i have immobile relatives in city i.) Our analysis goes through because each special city c_i^* competes only for agents in group i, and does so only with normal cities. If i's value from $c_j^* \neq c_i^*$ is some intermediate $C' \in (0, C)$, the problem becomes more complicated, but the relevance of the forces we study does not vanish as $k \to \infty$. Thus, our model can be taken as an analysis of Tiebout competition in the presence of imperfect substitutability. A similar logic would apply if cities are ex ante identical but have ex post market power due to moving costs.[27]

A related extension would allow for the endogenous creation of organizations. Our analysis suggests that an agent far from a steady state is less likely to create an organization, or more likely to create a non-democratic one.

The organizations we model are simple: all members have the same voting power, decisions are made by majority rule, and votes are cast independently. Appendix E.1 discusses how to extend our analysis to allow for supermajority rules or other electoral rules that make an agent other than the median pivotal.[28] However, there is much unexplored complexity regarding the internal structure of organizations. Agents may have endogenous voting power (seniority); or they may engage in collective behavior by voting as a bloc, joining an organization in large numbers in order to change its policy, or threatening to leave *en masse* to extract concessions. These behaviors are not likely in the context of Tiebout competition, but may be so in other applications.

Organizations may also set up various barriers to entry or membership restrictions (or, in the case of cities, there may be moving costs). In the paper, we consider two extreme cases: one with completely free entry and exit (Section 2) and one in which the organization can choose its set of members at will (Section 5). In Appendix E.2, we present an extension allowing for (exogenous) positive entry and exit costs. Because in equilibrium agents enter and leave the organization at most once, this does not change the analysis much. Modeling endogenous entry costs that can be chosen separately from the organization's main policy, on the other hand, is much

[27] Moving costs and idiosyncratic preferences are suggested in Tiebout (1956) and Epple and Romer (1991), respectively, as forces preventing convergence to a Tiebout equilibrium in practice.

[28] Even in democracies, higher-income agents may wield more political power (Benabou, 2000; Jack and Lagunoff, 2006).

harder, as the state space becomes multidimensional. However, the forces we study will still be present as long as the organization cannot perfectly control both its payoff-relevant policy and its membership (if it can, we are back in the world of a fixed decision-maker).

Our analysis abstracts away from history-dependent strategies by focusing on Markovian equilibria. In Appendix E.3, we show that Non-Markovian equilibria can support a large number of outcomes, but that several reasonable refinements select only Markovian equilibria. In particular, any equilibrium obtainable as a limit of discrete policy-space equilibria must be Markovian.

Finally, we do not explicitly model the organization's voting process. One way to interpret our results is that they will hold whenever the organization's collective decision-making process leads to Condorcet-winning policies being chosen. In Appendix E.4, we discuss possible microfoundations of this modeling assumption. In particular, the MVEs we study are Markov Perfect Equilibria of a game in which, in each round of voting, there are two short-lived, office-motivated candidates engaging in Downsian competition. However, not all political processes are so well-behaved. For instance, if the organization has more than two candidates running for office, or it is run by a deliberative decision-making process, then the Condorcet winner may not win.[29] In addition, leaders typically have some agency in practice. If they are long-lived and have heterogeneous appeal, or are policy-motivated, they may champion certain policies in an attempt to change the policy path, possibly permanently. Moreover, a politician may strategically push for policies that will attract a set of members predisposed to like her.[30]

Appendices B, C and D contain technical results. Appendix B contains the proofs of Propositions 6 and 7 and characterizes the case in which no CLS exists. Appendix C shows that the limit solution described in Appendix B exists for all m satisfying a genericity condition. Appendix D discusses the existence properties of equilibria other than 1Es (in particular, smooth equilibria and k-equilibria, a generalization of 1Es); calculates explicit equilibria for the case of linear m and quadratic utility; and gives an example of an MVE that is non-monotonic outside of $I(x^*)$.

[29]For example, Bouton and Gratton (2015) shows that Condorcet winners may lose in majority runoff elections with three candidates.

[30]Glaeser and Shleifer (2005) discusses the case of Mayor Curley of Boston, who used wasteful policies in an effort drive out rich citizens of English descent, as he was most popular among the poor Irish population.

A Proofs

Lemma 2. *Let $S = (x_0, x_1, \ldots)$, $T = (y_0, y_1, \ldots)$ be two policy paths, and let $I(S) = \cup_{n=0}^{\infty} I(x_n)$, $I(T) = \cup_{n=0}^{\infty} I(y_n)$. Suppose that $\sup x_n < \inf y_n$; $I(S)$, $I(T)$ are intervals; and $I(S) \cap I(T) \neq \emptyset$. Then there is α_0 such that agents in $[-1, \alpha_0)$ strictly prefer S to T, and agents in $(\alpha_0, 1]$ strictly prefer T to S.*

Proof. Let $\underline{x} = \inf x_n$, $\overline{x} = \sup x_n$, $\underline{y} = \inf y_n$, $\overline{y} = \sup y_n$. By assumption, $\underline{x} \leq \overline{x} < \underline{y} \leq \overline{y}$. Note that all agents $\alpha < \underline{x}$ strictly prefer S to T by A4 and A6; likewise, all $\alpha > \overline{y}$ strictly prefer T to S.

Let $W(\alpha) = U_\alpha(T) - U_\alpha(S)$. Note that W is continuous and $W(\underline{x}) < 0 < W(\overline{y})$. Hence there is some $\alpha_0 \in [\underline{x}, \overline{y}]$ for which $W(\alpha_0) = 0$. For any $\alpha \in [\underline{x}, \overline{y}]$, let $I_S^-(\alpha) = \sum_{n=0}^{\infty} \delta^n \mathbb{1}_{x_n \leq \alpha, u_\alpha(x_n) > 0}$, $I_S^+(\alpha) = \sum_{n=0}^{\infty} \delta^n \mathbb{1}_{x_n > \alpha, u_\alpha(x_n) > 0}$. Define $I_T^-(\alpha)$, $I_T^+(\alpha)$ analogously.

If $\alpha \in [\overline{x}, \underline{y}]$, then $I_S^-(\alpha) \geq I_T^-(\alpha) = 0 = I_S^+(\alpha) \leq I_T^+(\alpha)$, and $I_S^-(\alpha) + I_T^+(\alpha) > 0$ by the assumption that $I(S) \cap I(T) \neq \emptyset$. Then $W'(\alpha) > 0$ by A5.

If $\alpha \in [\underline{y}, \overline{y}]$, then $I_S^+(\alpha) = 0$, and one of the following must be true. If $I_S^-(\alpha) \leq I_T^-(\alpha)$, then $W(\alpha) > 0$ by A4, and moreover $W(\alpha') > 0$ for all $\alpha' \in [\alpha, \overline{y}]$. If $I_S^-(\alpha) > I_T^-(\alpha)$, then $W'(\alpha) > 0$ by A2 and A5. Similarly, for each $\alpha \in [\underline{x}, \overline{x}]$, either $W'(\alpha) > 0$ or $W(\alpha') < 0$ for all $\alpha' \in [\underline{x}, \alpha]$.

In general, then, we can find thresholds z_0, z_1 such that $\underline{x} \leq z_0 \leq \overline{x} < \underline{y} \leq z_1 \leq \overline{y}$; $W'(\alpha) > 0$ for all $\alpha \in [z_0, z_1]$; $W(\alpha) < 0$ for all $\alpha \in [\underline{x}, z_0)$; and $W(\alpha) > 0$ for all $\alpha \in (z_1, \overline{y}]$. Hence W can vanish at most at one point, so α_0 is unique. □

Lemma 3. *In any MVE s, for any y, $I(y) \cap I(s(y))$ has positive measure. Hence $I(S(y))$ is an interval for all y.*

Proof. Suppose WLOG $y < s(y)$. We argue that $y + d_y^+ > s(y) - d_{s(y)}^-$. Suppose this is false; then all voters in $I(y)$ get utility 0 from policy $s(y)$. If $E = \{\alpha \in I(y) : U_\alpha(S(s(y))) > 0\}$ is a *strict* majority of $I(y)$, this leads to a contradiction, as all of E would strictly prefer $S(s^2(y))$ to $S(s(y))$. Let $D = \{\alpha \in I(y) : U_\alpha(S(s(y))) \geq U_\alpha(S(y))\} \subseteq E$. Since $S(s(y))$ is a Condorcet winner in $I(y)$, D is a majority in $I(y)$. If $I(y) \subseteq D$ we are done. If not, $\exists \alpha_0 \in I(y) - D$. By the continuity of $U_\alpha(S(s(y)))$, $\exists \alpha_1$ such that $0 < U_{\alpha_1}(S(s(y))) < U_{\alpha_1}(S(y))$, and this inequality holds in some neighborhood $(\alpha_1 - \varepsilon, \alpha_1 + \varepsilon)$. Hence $E - D$ has positive measure and E is a strict majority of $I(y)$. □

Corollary 2. *In any MVE, let $S = S(y)$ for some $y < x$ and $T = (x, x, \ldots)$, with $\sup(S) \leq x$. Then there is $\alpha_0 \leq x$ such that agents in $[-1, \alpha_0)$ strictly prefer S to T, and agents in $(\alpha_0, 1]$ strictly prefer T to S.*

Proof. If $I(S) \cap I(x) \neq \emptyset$, this follows directly from Lemmas 2 and 3. If not, then all $\alpha \geq x - d_x^-$ strictly prefer T to S and all $\alpha \leq \sup I(S)$ strictly prefer S to T. Let α_0' be such that $u_{\alpha_0'}(y) = u_{\alpha_0'}(x) < 0$. If $\alpha_0' \in (\sup I(S), x - d_x^-)$ then take $\alpha_0 = \alpha_0'$. If $\alpha_0' \leq \sup I(S)$ then take $\alpha_0 = I(S)$. If $\alpha_0' \geq x - d_x^-$ take $\alpha_0 = x - d_x^-$. $\square$

Proof of Proposition 1. Suppose $S(y)$ is not monotonic. For this proof, denote $s_k = s^k(y)$, $S_k = S(s_k)$, $I_k = I(s_k)$, $\underline{y} = \inf(S(y))$ and $\overline{y} = \sup(S(y))$. For brevity, we will say α prefers a policy x to a path S if she prefers $(x, x, \ldots)$ to S. There are two cases:

Case 1: $S(y)$ attains $\underline{y}$ or $\overline{y}$, i.e., $\exists k \in \mathbb{N}$ such that $s_k = \overline{y}$ or $s_k = \underline{y}$. Suppose WLOG the former. Then there is a $k \in \mathbb{N}$ such that $s_{k-1} < \overline{y}$, $s_k = \overline{y}$ and $s_{k+1} < \overline{y}$.[31] Consider the decision made by voters in I_{k-1} and in I_k. Since S_k is the Condorcet winner in I_{k-1}, a majority of I_{k-1} prefer it to S_{k+1}. At the same time, S_{k+1} is Condorcet-winning in I_k, so a majority of I_k prefer S_{k+1} to S_k. Let $A = (s_{k-1} - d_{k-1}^-, s_k - d_k^-)$, $B = (s_k - d_k^-, s_{k-1} + d_{k-1}^+)$, $C = (s_{k-1} + d_{k-1}^+, s_k + d_k^-)$. Note that α prefers S_k to S_{k+1} iff he prefers s_k to S_{k+1}. Apply Corollary 2. If $\alpha_0 \in C$, all voters in $A \cup B$ strictly prefer S_{k+1} to S_k, a contradiction. If $\alpha_0 \in B$, all voters in A strictly prefer S_{k+1} to S_k and all voters in C strictly prefer S_k to S_{k+1}, a contradiction.

Case 2: $S(y)$ never attains its infimum nor its supremum. Then there must be a subsequence $(s_{k_i})_i$ such that $s_{k_i} \underset{i \to \infty}{\longrightarrow} \overline{y}$. Construct a sub-subsequence $s_{k_{i_j}}$ such that $s_{k_{i_j}} \underset{j \to \infty}{\longrightarrow} \overline{y}$ and $s_{k_{i_j}-1} \underset{j \to \infty}{\longrightarrow} s_*^{-1}$ for some limit $s_*^{-1} \leq \overline{y}$. (We can do this because all the s_k are in $[-1, 1]$, which is compact.) Iterating this, construct a nested list of subsequences $((s_{k_{im}})_i)_m$ such that k_{im} is increasing in i for each m; $K_m = \{k_{im} : i \geq 0\} \supseteq K_{m'}$ for $m < m'$; and, for each m, $s_{k_{im}+r} \underset{i \to \infty}{\longrightarrow} s_*^r$ for any $r \in \{-m, \ldots, m\}$, where $s_*^0 = \overline{y}$. Let $g_i = k_{ii}$. Then $(s_{g_i})_i$ is a subsequence of $(s_k)_k$ such that $s_{g_i+r} \underset{i \to \infty}{\longrightarrow} s_*^r$ for any $r \in \mathbb{Z}$. We now consider four sub-cases.

Case 2.1: Suppose $s_*^r < \overline{y}$ for some $r < 0$ and for some $r' > 0$, and let $\underline{r} < 0 < \overline{r}$ be the numbers closest to 0 satisfying these conditions. Consider the decisions made by $I_{g_i+\underline{r}}$ and $I_{g_i+\overline{r}-1}$, for high i. In the limit, they imply that a majority in $I(s_*^{\underline{r}})$ prefers $\overline{y}$ to $\tilde{S}(s_*^{\overline{r}})$, while a majority in $I(\overline{y})$ prefers $\tilde{S}(s_*^{\overline{r}})$ to $\overline{y}$ (denoting $\tilde{S}(s_*^{\overline{r}}) = (s_*^{\overline{r}}, s_*^{\overline{r}+1}, \ldots)$). As in Case 1, this contradicts Corollary 2.

[31] The same argument would apply if $s^k(y) = \ldots = s^{k+m}(y) > s^{k-1}(y), s^{k+m+1}(y)$.

Case 2.2: Suppose $s_*^r < \overline{y}$ for some $r < 0$ but never for $r > 0$. Take $\underline{r}$ maximal, so $s_*^{\underline{r}} < \overline{y}$ and $s_*^r = \overline{y}\ \forall r > \underline{r}$. Fix $0 < \nu < \overline{y} - s_*^{\underline{r}}$. For each i, let $r'(i)$ be such that $s_{g_i + r'(i)}$ is the first element of the sequence $(s_k)_k$ after $s_{g_i+\underline{r}}$ that is weakly smaller than $s_*^{\underline{r}} + \nu$. Construct a subsequence $(s_{g_{i_j}})_j$ such that $s_{g_{i_j}+r'(i_j)+l} \to s_{**}^l$ for $l \geq -1$ (in particular $s_{**}^{-1} \geq s_*^{\underline{r}} + \nu \geq s_{**}^0$). Now compare the decisions made by $I(s_{g_i+\underline{r}})$ and $I(s_{g_i+r'(i)-1})$. In the limit, they imply that a weak majority in $I(s_*^{\underline{r}})$ prefers $\overline{y}$ to $\tilde{S}(s_{**}^0)$, while a weak majority in $I(s_{**}^{-1})$ prefers the opposite (here $\tilde{S}(s_{**}^0) = (s_{**}^0, s_{**}^1, \ldots)$). This contradicts Corollary 2.

Case 2.3: Suppose $s_*^r < \overline{y}$ for some $r > 0$ but never for $r < 0$. Take $\overline{r}$ minimal, so $s_*^{\overline{r}} < \overline{y}$ and $s_*^r = \overline{y}\ \forall r < \overline{r}$. Fix $0 < \nu < \overline{y} - s_*^{\overline{r}}$. Let $r'_\nu(i)$ be such that $s_{g_i+r'_\nu(i)}$ is the last element before s_{g_i} that is weakly smaller than $\overline{y} - \nu$. Clearly $r'_\nu(i) \underset{i\to\infty}{\longrightarrow} -\infty$.

Consider the choice made by $I(s_{g_i+\overline{r}-1})$. In the limit, a majority in $I(\overline{y})$ prefers $(s_*^{\overline{r}}, s_*^{\overline{r}+1}, \ldots)$ over $\overline{y}$. Apply Corollary 2. Clearly $\alpha_0 < \overline{y}$, so $m(\overline{y}) \leq \alpha_0 < \overline{y}$. As m is strictly increasing, $m^k(\overline{y})$ is strictly decreasing in k and converges to a limit $\tilde{m}$; moreover, $m(y) < y$ for all $y \in (\tilde{m}, \overline{y}]$. Call $g_i + r'_\nu(i) = h_{i\nu}$ and let $s_*^\nu = \liminf_{i\to\infty} s_{h_i\nu}$. Let $s_{**} = \liminf_{\nu\to 0} s_*^\nu$. If $s_{**} < \overline{y}$, take a sequence of $\nu, h_{i\nu}$ such that $s_{h_{i\nu}} \to s_{**}$. By construction $s_{h_{i\nu}+l} \geq \overline{y} - \nu$ for $l = 1, \ldots, L$ for L arbitrarily large as $\nu \to 0$, $h_{i\nu} \to \infty$. Then, in the limit, $\overline{y}$ is a Condorcet winner in $I(s_{**})$; in particular, a majority prefers $\overline{y}$ to $(s_*^{\overline{r}}, s_*^{\overline{r}+1}, \ldots)$, which contradicts Corollary 2.

If $s_{**} = \overline{y}$ we must work away from the limit. Take $\varepsilon > 0$ such that $(\overline{y} - d_{\overline{y}}^- + \varepsilon, \overline{y} - \varepsilon) \subseteq I(\overline{y} - \upsilon)$ is a strict majority of $I(\overline{y} - \upsilon)$ for all $0 < \upsilon \leq \varepsilon$.[32] Take a fixed $\nu' < \varepsilon$; a $\nu < \nu'$ such that $s_*^\nu \geq \overline{y} - \nu'$; and a subsequence s_{f_i} of $s_{h_{i\nu}}$ such that $s_{f_i} \to s_*^\nu$. Let M_i be the largest integer such that $s_{f_i+l} \in (\overline{y} - \nu, \overline{y})$ for $l = 1, \ldots, M_i$ and K_i the set of $l \in 1, \ldots, M_i$ such that $s_{f_i+l} \in (\overline{y} - \frac{\nu}{2}, \overline{y})$. Let $k_i = \min(K_i)$. By construction, $M_i, |K_i| \to \infty$. Then, for $\alpha \in (\overline{y} - d_{\overline{y}}^- + \varepsilon, \overline{y} - \varepsilon)$,

$$\frac{1}{1-\delta} u_\alpha(s_{f_i}) - U_\alpha(S(s_{f_i+1})) = \sum_{t \in K_i} \delta^{t-1} \left(u_\alpha(s_{f_i}) - u_\alpha(s_{f_i+t})\right) +$$

$$+ \sum_{M_i \geq t \notin K_i} \delta^{t-1} \left(u_\alpha(s_{f_i}) - u_\alpha(s_{f_i+t})\right) + \sum_{t > M_i} \delta^{t-1} \left(u_\alpha(s_{f_i}) - \mathbb{1}_{\alpha \in I(s_{f_i+t})} u_\alpha(s_{f_i+t})\right)$$

$$\geq \delta^{k_i-1} \left(u_\alpha(s_{f_i}) - u_\alpha(s_{f_i+k_i})\right) + 0 - \frac{\delta^{M_i}}{1-\delta} C = \delta^{k_i-1} \left(u_\alpha(s_{f_i}) - u_\alpha(s_{f_i+k_i}) - \frac{\delta^{|K_i|}}{1-\delta} C\right)$$

where $C = \max_\alpha u_\alpha(\alpha)$. Note that $u_\alpha(s_{f_i}) - u_\alpha(s_{f_i+k_i}) \geq u_{s_{f_i}}(s_{f_i}) - u_{s_{f_i}}(s_{f_i+k_i}) +$

[32] ε exists because $m(\overline{y}) < \overline{y}$ and $y - d_y^-$, $y + d_y^+$ are continuous in y.

$M'(s_{f_i} - \alpha)(s_{f_i+k_i} - s_{f_i}) \geq M'(s_{f_i} - \overline{y} + \varepsilon)\frac{\nu}{2}$, which converges to $M'(s_*^\nu - \overline{y} + \varepsilon)\frac{\nu}{2} \geq M'\frac{\nu}{2}(\varepsilon - \nu') > 0$ as $i \to \infty$. On the other hand, $\delta^{|K_i|} \to 0$ as $i \to \infty$. Hence all $\alpha \in (\overline{y} - d_{\overline{y}}^- + \varepsilon, \overline{y} - \varepsilon)$ prefer s_{f_i} to $S(s_{f_i+1})$ for high i, so $S(s_{f_i+1})$ is not a Condorcet winner in $I(s_{f_i})$, a contradiction.

Case 2.4: $s_*^r = \overline{y}$ for all r. In other words, the sequence spends arbitrarily long times near $\underline{y}$ and $\overline{y}$ (if not true for both boundaries, one of the former cases applies). We first prove the following claim: $m(y) = y$ for all $y \in [\underline{y}, \overline{y}]$.

Take any $y_0 \in (\underline{y}, \overline{y})$. Take a sequence $(h_i)_i$ such that, for each i, s_{h_i} is the last element of the sequence $(s_k)_k$ before s_{g_i} such that $s_k \leq y_0$. Intuitively, s_{h_i} is the last element of the sequence below y_0 before the sequence goes near $\overline{y}$ for a long time. Take a diagonal subsequence (s_{k_i}) of (s_{h_i}) such that s_{k_i+l} has a limit s_{**}^l for all i. Clearly $s_{**}^0 \leq y_0$ and $s_{**}^l \geq y_0$ for all $l > 0$.

Consider the choice made by $I(s_{k_i})$. If $s_{**}^0 < y_0$, in the limit, a majority in $I(s_{**}^0)$ prefers $(s_{**}^1, s_{**}^2, \ldots)$ over s_{**}^0. Apply Corollary 2. Clearly $m(s_{**}^0) \geq \alpha_0 > s_{**}^0$, so $u_{m(s_{**}^0)}(s_{**}^0) \leq u_{m(s_{**}^0)}(y_0)$. If $s_{**}^0 = y_0$, then $m(y_0) < y_0$ leads to a contradiction by an analogous argument as in Case 2.3, so we must have $m(y_0) \geq y_0$. Conversely, considering sequences going near $\underline{y}$ for arbitrarily long, we obtain that either $m(y_0) \leq y_0$ or there is $\tilde{s}_{**}^0 > y_0$ such that $m(\tilde{s}_{**}^0) < \tilde{s}_{**}^0$ and $u_{m(\tilde{s}_{**}^0)}(\tilde{s}_{**}^0) \leq u_{m(\tilde{s}_{**}^0)}(y_0)$.

For each $y \in (\underline{y}, \overline{y})$ such that $y \neq m(y)$, define $\hat{y} \neq y$ to be such that $u_{m(y)}(y) = u_{m(y)}(\hat{y})$. Take y_0 such that $|y_0 - \hat{y}_0|$ is maximal. WLOG $m(y_0) < y_0$, so there is $s_{**}^0 < y_0$ such that $m(s_{**}^0) > s_{**}^0$ and $u_{m(s_{**}^0)}(s_{**}^0) \leq u_{m(s_{**}^0)}(y_0)$. Since $m(y_0) > m(s_{**}^0)$, $u_{m(y_0)}(s_{**}^0) < u_{m(y_0)}(y_0)$, so $\hat{y}_0 > s_{**}^0$; but $\hat{s}_{**}^0 \geq y_0$. Hence $|s_{**}^0 - \hat{s_{**}^0}| > |\hat{y}_0 - y_0|$, a contradiction.

For the case where $m(y) = y$ for all $y \in [\underline{y}, \overline{y}]$, we use the following Lemma.

Lemma 4. *Let $S = (y, y, \ldots)$, and let $T = (x_n)_n \neq S$. If x and x' both prefer T to S, and $x < y < x'$, then $x \notin I(x')$ or $x' \notin I(x)$.*

Proof. Suppose for that $x \in I(x')$ and $x' \in I(x)$. It is enough to check the case where T is contained in $[x, x']$: if not, define a path $(\tilde{x}_n)_n$ by $\tilde{x}_n = \min(\max(x_n, x), x')$. Then $(\tilde{x}_n)_n$ is contained in $[x, x']$ and is weakly better for both x and x' than T.

By assumption, both x and x' derive non-negative utility from all elements of T. Let $\overline{x} = (1 - \delta)\sum_n \delta^n x_n$, and $T'' = (\overline{x}, \overline{x}, \ldots)$. If $T'' \neq T$, both x and x' strictly prefer T'' to T by Jensen's inequality and A4. Hence they both strictly prefer $\overline{x}$ to y, a contradiction. If $T'' = T$, $\overline{x} \neq y$ and both agents prefer $\overline{x}$ to y, a contradiction. $\square$

Take $\epsilon > 0$, $\nu > 0$ small and $y_0 = \underline{y} + \epsilon$. Construct s_{k_i} as before. It follows from previous arguments that $s^0_{**} = y_0$. For all i, a majority in $I(s_{k_i})$ must prefer $S(s_{k_i+1})$ over s_{k_i}. Since s_{k_i} strictly prefers s_{k_i} over $S(s_{k_i+1})$ and $s_{k_i} = m(s_{k_i})$, this can only happen if there are voters both above and below s_{k_i} who prefer $S(s_{k_i+1})$. Let $y'_i < s_{k_i} < y''_i$ be the closest voters to s_{k_i} who weakly prefer $S(s_{k_i+1})$, and denote $y'_i - (s_{k_i} - d^-_{s_{k_i}}) = \eta'_i$, $y''_i - s_{k_i} = \eta''_i$. Note that $\eta'_i, \eta''_i \underset{i\to\infty}{\longrightarrow} 0$.[33] In addition, $y''_i - d^-_{y''_i} > y'_i$; otherwise we obtain a contradiction as in Lemma 4. Let $\tilde{y}_i$ be such that $\tilde{y}_i - d^-_{\tilde{y}_i} = y'_i$.

Given the path $T^i = S(s_{k_i+1})$ construct T'^i as follows. If $T^i_j \geq y''_i + \nu$, $T'^i_j = y''_i + \nu$. If $y''_i + \nu > T^i_j \geq \tilde{y}_i$, $T'^i_j = y''_i$. If $\tilde{y}_i > T^i_j \geq s_{k_i}$, $T'^i_j = z_i = \frac{\sum_{\tilde{y}_i > T^i_j \geq s_{k_i}} \delta^j T^i_j}{\sum_{\tilde{y}_i > T^i_j \geq s_{k_i}} \delta^j}$. If $s_{k_i} > T^i_j$, $T'^i_j = v_i = \frac{\sum_{s_{k_i} > T^i_j} \delta^j T^i_j}{\sum_{s_{k_i} > T^i_j} \delta^j}$. Then both y'_i and y''_i weakly prefer T'^i over T^i. Moreover, T'^i is a linear combination of at most four policies; by an abuse of notation, $T'^i = \omega^i_1[y''_i] + \omega^i_2[y''_i + \nu] + \omega^i_3[z_i] + \omega^i_4[v_i]$ with $\sum_j \omega^i_j = 1$. In addition, since $S(s_{k_i+1})$ spends a long time near $\overline{y}$ (hence above $y''_i + \nu$) before going back under s_{k_i}, $\frac{\omega^i_4}{\omega^i_2} \underset{i\to\infty}{\longrightarrow} 0$.[34]

Finally, take $0 < \omega^i_5 \leq \omega^i_4$ such that $\omega^i_3 z_i + \omega^i_5 v_i = (\omega^i_3 + \omega^i_5) s_{k_i}$,[35] and construct $T'''^i = \omega^i_1[y''_i] + \omega^i_2[y''_i + \nu] + (\omega^i_3 + \omega^i_5)[s_{k_i}] + (\omega^i_4 - \omega^i_5)[v_i]$, $T''^i = w^i_1[y''_i] + w^i_2[y''_i + \nu] + w^i_3[v_i]$ where $w^i_1 = \frac{\omega^i_1}{\omega^i_1 + \omega^i_2 + \omega^i_4 - \omega^i_5}$, $w^i_2 = \frac{\omega^i_2}{\omega^i_1 + \omega^i_2 + \omega^i_4 - \omega^i_5}$, $w^i_3 = \frac{\omega^i_4 - \omega^i_5}{\omega^i_1 + \omega^i_2 + \omega^i_4 - \omega^i_5}$ and $\frac{w^i_3}{w^i_2} \underset{i\to\infty}{\longrightarrow} 0$. Then both y'_i and y''_i weakly prefer T''^i over T^i and hence over s_{k_i}. Then, for some $C, c > 0$,

$$C w^i_3 \geq w^i_3 u_{y'_i}(v_i) \geq u_{y'_i}(s_{k_i}) = u_{y'_i}(s_{k_i}) - u_{y'_i - \eta'_i}(s_{k_i}) = \eta'_i \frac{\partial u_{\tilde{\alpha}}(s_{k_i})}{\partial \alpha} \geq c\eta'_i$$

$$u_{y''_i}(s_{k_i}) \leq w^i_1 u_{y''_i}(y''_i) + w^i_2 u_{y''_i}(y''_i + \nu) + w^i_3 u_{y''_i}(v_i) \leq (w^i_1 + w^i_2) u_{y''_i}\left(y''_i + \frac{w^i_2 \nu}{w^i_1 + w^i_2}\right) + w^i_3 u_{y''_i}(s_{k_i})$$

$$u_{y''_i}(y''_i - \eta''_i) = u_{y''_i}(s_{k_i}) \leq u_{y''_i}\left(y''_i + \frac{w^i_2 \nu}{w^i_1 + w^i_2}\right) \leq u_{y''_i}\left(y''_i + w^i_2 \nu\right)$$

As A2 and A4 imply $u_\alpha(\alpha) - u_\alpha(\alpha - x) \in \left[\frac{M'}{2}x^2, \frac{M}{2}x^2\right]$, this means

$$\frac{M}{2}(\eta''_i)^2 \geq \frac{M'}{2}(w^i_2 \nu)^2 \Longrightarrow \frac{\eta''_i}{\eta'_i} \geq \sqrt{\frac{M'}{M}} \frac{c}{C} \frac{w^i_2}{w^i_3} \nu$$

Since (y'_i, y''_i) cannot be a strict majority in $I(s_{k_i})$, we must have $F(y''_i) - F(s_{k_i}) \leq$

[33] $\eta'_i \to 0$ by a similar argument to Case 2.3. Then, if η''_i did not converge to zero, (y'_i, y''_i) would be a strict majority in $I(s_{k_i})$ for large i, a contradiction.

[34] For these arguments to work, we take ν, ϵ small enough that $y_0 + \nu < \overline{y}$ and $u_{y_0}(\underline{y}) > 0$.

[35] If this is not possible then y'_i could not have preferred T''^i over s_{k_i}, a contradiction.

$F(y_i') - F(s_{k_i} - d^-_{s_{k_i}})$ for all i. But this is impossible as $\frac{f(x)}{f(x')}$ is bounded and $\frac{\eta_i''}{\eta_i'} \underset{i\to\infty}{\longrightarrow} \infty$, a contradiction. $\square$

Proof of Proposition 2. Suppose $m(y) = y$ and $s(y) \neq y$; WLOG $s(y) < y$. A majority in $I(y)$ must prefer $S(s(y))$ to $S(y)$, i.e., they must prefer $S(s(y))$ to y. By Proposition 1, $s^k(y) \leq s(y)$ for all k. But then, for small enough $\epsilon > 0$, all agents in $(y - \epsilon, y + d^+_y)$ strictly prefer y to $S(s(y))$, a contradiction.

If $m(y) \neq y$, suppose WLOG that $m(y) < y$. If $s(y) > y$ then $s^k(y) \geq s(y) > y$ for all k, so all voters in $(y - d^-_y, y)$ (a strict majority in $I(y)$) strictly prefer y to $S(s(y))$, a contradiction. Hence $s(y) \leq y$. On the other hand, suppose $s(y) < m^*(y)$. Note that $m^*(y) < m(y)$; $m(m^*(y)) = m^*(y)$; $s^k(y) \leq s(y)$ for all k; and choosing $m^*(y)$ leads to the policy path $(m^*(y), m^*(y), \ldots)$ by the previous case. Then, for small enough $\epsilon > 0$, all voters in $(m(y) - \epsilon, y + d^+_y)$ prefer $S(m^*(y))$ over $S(s(y))$, a contradiction. Hence $s(y) \geq m^*(y)$. Next, suppose $s(y) = m^*(y)$ and consider $T = (m(y), s(m(y)), \ldots)$. Since T is contained in $[m^*(y), m(y)]$ and $T_1 = m(y) > m^*(y)$, all voters in $(m(y) - \epsilon, y + d^+_y)$ for small $\epsilon > 0$ strictly prefer T over $S(s(y))$, a contradiction. Hence $s(y) > m^*(y)$.

We now show that, if the MVT holds on $[m^*(y), y]$, then $s(y) < y$. Suppose that $s(y) = y$. There must be ϵ_0 such that $s(y - \epsilon) < y - \epsilon$ for all $0 < \epsilon < \epsilon_0$ (otherwise, $m(y)$ would prefer the constant path $(y - \epsilon, y - \epsilon, \ldots)$ to $(y, y, \ldots)$ for ϵ small enough).

Let $s_-(y) = \liminf_{\epsilon\to 0} s(y - \epsilon) \in [m^*(y), y]$. There are two cases: $s_-(y) = y$ and $s_-(y) < y$. If $s_-(y) = y$, then $s^k(x) \to y$ as $x \to y$ for all k. For all $x \in (y - \epsilon_0, y)$, $m(x)$ must prefer $S(s(x))$ to x. That is, denoting $W(x) = (1 - \delta)U_{m(x)}(S(s(x))) - u_{m(x)}(x)$, we must have $W(x) \geq 0$. Equivalently

$$(1 - \delta) \sum_{t=0}^{k_x} \delta^t u_{m(x)} \left(s^{t+1}(x)\right) - u_{m(x)}(x) \geq 0,$$

where $k_x = \max\{k : m(x) \in I(s^k(x))\} - 1$. (Note that $k_x \underset{x\to y}{\longrightarrow} \infty$.) By the envelope theorem,

$$W'(x) = (1 - \delta)\frac{\partial}{\partial\alpha} U_{m(x)}(S(s(x)))m'(x) - \frac{d}{dx} u_{m(x)}(x) =$$
$$= \sum_{t=0}^{k_x} (1 - \delta)\delta^t \left(\frac{\partial}{\partial\alpha} u_{m(x)} \left(s^{t+1}(x)\right) - \frac{\partial u}{\partial\alpha}\right) m'(x) - (1 - \delta^{k_x+1})\frac{\partial u}{\partial x} - \delta^{k_x+1}\frac{du}{dx}$$

$$\geq \sum_{t=0}^{k_x} \delta^t \left(-M(x - s^{t+1}(x))\right) m'(x) - (1 - \delta^{k_x+1}) \frac{\partial u}{\partial x} - \delta^{k_x+1} \frac{du}{dx}$$
$$\xrightarrow[x \to y]{} -\frac{\partial}{\partial x} u_{m(y)}(y) > 0,$$

where u stands for $u_{m(x)}(x)$ unless otherwise noted. Thus $W(x) \geq 0$ and $W'(x) > 0$ for all $x \in (y - \epsilon_1, y)$, whence $W(y) > 0$, which contradicts $s(y) = y$.

If $s_-(y) < y$, let $(y_n)_n$ be a sequence such that $y_n < y \; \forall n$, $y_n \to y$ and, for all t, $s^t(y_n)$ converges to a limit s_t as $n \to \infty$ (in particular $s_1 = s_-(y)$). By construction, $m(y)$ must prefer y to $S(s(y_n))$ for all n. We now aim to show that

$$\frac{U_{m(y)}(S(s(y_n))) - \frac{1}{1-\delta} u_{m(y)}(y)}{y - y_n} \xrightarrow[n \to \infty]{} 0.$$

$m(y_n)$ prefers $S(s(y_n))$ to y for all n. By continuity, $m(y)$ is indifferent between y and $(s_t)_t$. Moreover, $m(y_n)$ prefers $S(s(y_n))$ to all other $S(s(y_{n'}))$, hence to $(s_t)_t$. Thus

$$0 \geq U_{m(y)}(S(s(y_n))) - \frac{1}{1-\delta} u_{m(y)}(y) = U_{m(y)}(S(s(y_n))) - U_{m(y)}((s_t)_t) \geq$$
$$\geq U_{m(y)}(S(s(y_n))) - U_{m(y)}((s_t)_t) + U_{m(y_n)}((s_t)_t) - U_{m(y_n)}(S(s(y_n))) = \sum_{t=0}^{\infty} \delta^t A_{tn},$$

where, denoting $v_\alpha(x) = \max(u_\alpha(x), 0)$,

$$A_{tn} = v_{m(y)}(s^{t+1}(y_n)) - v_{m(y)}(s_{t+1}) + v_{m(y_n)}(s_{t+1}) - v_{m(y_n)}(s^{t+1}(y_n)).$$

Let $B_{tn} = \frac{A_{tn}}{y - y_n}$. Then it is sufficient to show that B_{tn} is uniformly bounded (that is, $\exists \overline{B}$ such that $|B_{tn}| \leq \overline{B}$ for all t, n) and that, for all t, $\liminf_{n \to \infty} B_{tn} \geq 0$.

We first prove the boundedness. Using that $|\max(a, 0) - \max(b, 0)| \leq |a - b|$,

$$A_{tn} \leq |A_{tn}| \leq |u_{m(y)}(s^{t+1}(y_n)) - u_{m(y_n)}(s^{t+1}(y_n))| + |u_{m(y_n)}(s_{t+1}) - u_{m(y)}(s_{t+1})| \leq$$
$$\leq 2\overline{m}' \max_{\alpha, x} \left[\frac{\partial u_\alpha(x)}{\partial \alpha}\right] (y - y_n),$$

where $\overline{m}' = m'(x)$. Next, we prove that $\liminf_{n \to \infty} B_{tn} \geq 0$. There are four cases. First, if $u_{m(y)}(s_{t+1}) > 0$, then there is n_0 such that for all $n \geq n_0$ $u_{m(y)}(s^{t+1}(y_n))$, $u_{m(y_n)}(s_{t+1})$, $u_{m(y_n)}(s^{t+1}(y_n)) > 0$. For all such n, by A2, there is $\tilde{M}_{tn} \in [M', M]$

such that $A_{tn} = \tilde{M}_{tn}\left(s^{t+1}(y_n) - s_{t+1}\right)(m(y) - m(y_n))$, so $|B_{tn}| \xrightarrow[n\to\infty]{} 0$. Second, if $u_{m(y)}(s_{t+1}) < 0$, then for all large enough n $A_{tn} = 0$. Third, if $u_{m(y)}(s_{t+1}) = 0$ and $s^{t+1}(y_n) \geq s_{t+1}$, all the terms are positive and we use the same argument as in case 1. Fourth, if $u_{m(y)}(s_{t+1}) = 0$ and $s^{t+1}(y_n) < s_{t+1}$, then $u_{m(y_n)}(s_{t+1}) > u_{m(y_n)}(s^{t+1}(y_n))$ and $v_{m(y)}(s_{t+1}) = v_{m(y)}(s^{t+1}(y_n)) = 0$, so $B_{nt} \geq 0$.

Consider now the possibility of $m(y)$ choosing $S(y_n)$ instead. We can see that

$$
\begin{aligned}
&(1-\delta)U_{m(y)}(S(y_n)) - u_{m(y)}(y) = \\
&= (1-\delta)\left(u_{m(y)}(y_n) - u_{m(y)}(y)\right) + \delta\left((1-\delta)U_{m(y)}(S(s(y_n))) - u_{m(y)}(y)\right) \\
&= (1-\delta)(y_n - y)\frac{\partial u_{m(y)}(\tilde{y})}{\partial x} + o(y - y_n)
\end{aligned}
$$

for some $\tilde{y} \in [y_n, y]$. Since $\frac{\partial u_{m(y)}(\tilde{y})}{\partial x} \xrightarrow[n\to\infty]{} \frac{\partial u_{m(y)}(y)}{\partial x} < 0$, the above expression is positive for high n, so $s(y) = y$ is not optimal for $m(y)$, a contradiction.

Finally, we show that $s^k(y)$ converges to $m^*(y)$. Since $s^k(y) \in [m^*(y), y]$ for all y and the sequence $(s^k(y))_k$ is decreasing, it has a limit $s^* \in [m^*(y), y)$. Suppose $s^* > m^*(y)$. Then $m(s^*) < s^*$, so there is k_0 such that $m(s^k(y)) < s^*$ for all $k \geq k_0$. For such k, $m(s^k(y))$ would strictly prefer $S(s^{k+2}(y))$ to $S(s^{k+1}(y))$, a contradiction. □

Proof of Corollary 1. Let $x_i^* < x_{i+1}^*$ be consecutive fixed points of m. Since m is continuous, either $m(y) > y$ $\forall y \in (x_i^*, x_{i+1}^*)$ or $m(y) < y$ for all such y. The first case implies $m'(x_i^*) \geq 1$ and $m'(x_{i+1}^*) \leq 1$, and vice versa; since $m'(x_j^*) \neq 1$, these inequalities are strict, which implies that the intervals must alternate.

A fixed point of m is stable if $m'(x^*) < 1$ and unstable if $m'(x^*) > 1$ (see e.g. Elaydi (2005), Chapter 1.5). Since $m(-1) > -1$ and $m(1) < 1$, x_1^* and x_n^* are both stable, and stable and unstable fixed points alternate in between. □

Proof of Proposition 3. We first prove the monotonicity. Fix $\epsilon > 0$ small. Let $x < y \in [x^*, x^* + \epsilon]$, where $m(x^*) = x^*$, $m(x^{**}) = x^{**}$, $m(y) < y$ for all $y \in (x^*, x^{**})$ and $\epsilon < x^{**} - x^*$. Call $s^k(x) = x_k$, $s^k(y) = y_k$ and suppose $x_1 > y_1$. Then $S(x_1)$ is preferred to $S(y_1)$ by a majority in $I(x)$, and the opposite happens in $I(y)$. Since all agents in $I(y) - I(x) = (x + d_x^+, y + d_y^+]$ prefer $S(x_1)$ due to A6, there must also be $z_0 \in I(x) - I(y) = [x - d_x^-, y - d_y^-)$ that prefers $S(x_1)$ (in fact there must be enough of them, but we only need one).

Let l be such that $x_i > y_i$ for $i = 1, 2, \ldots, l$ but not for $i = l+1$. If $x_{l+1} = y_{l+1}$ (and hence $S(x_{l+1}) = S(y_{l+1})$) we have a contradiction, as any $z_0 \in I(x) - I(y)$ would prefer

$S(y_1)$ to $S(x_1)$ pointwise. By a similar argument, there must be $z_l \in I(y_l) - I(x_l)$ that prefers $S(y_{l+1})$ to $S(x_{l+1})$. If $x_i < y_i$ for *all* $i \geq l+1$ this also yields a contradiction, as any such z_l would prefer $S(x_{l+1})$ over $S(y_{l+1})$ pointwise. More generally, if the ordering between x_l and y_l only changes a finite number of times, we can obtain a contradiction by looking at the last change. The only case left to consider is if there are arbitrarily high i's and j's for which $x_i > y_i$ and $x_j < y_j$. If so, note that

$$0 \leq U_{z_0}(S(x_1)) - U_{z_0}(S(y_1)) = \sum_{t=0}^{l-1} \delta^t \left(u_{z_0}(x_{1+t}) - u_{z_0}(y_{1+t})\right) + \sum_{t \geq l} \delta^t \left(u_{z_0}(x_{1+t}) - u_{z_0}(y_{1+t})\right)$$

$$-\frac{\partial u_{z_0}(x)}{\partial x}|_{x^*} \sum_{t=0}^{l-1} \delta^t \left(x_{1+t} - y_{1+t}\right) \leq \sum_{t=0}^{l-1} \delta^t \left(u_{z_0}(y_{1+t}) - u_{z_0}(x_{1+t})\right) \leq \sum_{t \geq l} \delta^t \left(u_{z_0}(x_{1+t}) - u_{z_0}(y_{1+t})\right)$$

(Note that z_0 gets positive utility from all policies in $S(x_1)$ and $S(y_1)$.) Then

$$0 \leq U_{z_l}(S(y_{l+1})) - U_{z_l}(S(x_{l+1})) = \sum_{t \geq l} \delta^{t-l} \left(u_{z_l}(y_{1+t}) - u_{z_l}(x_{1+t})\right)$$

$$-\frac{\partial u_{z_0}(x)}{\partial x}|_{x^*} \sum_{t=0}^{l-1} \delta^t \left(x_{1+t} - y_{1+t}\right) \leq \sum_{t \geq l} \delta^t \left(u_{z_0}(x_{1+t}) - u_{z_l}(x_{1+t}) - u_{z_0}(y_{1+t}) + u_{z_l}(y_{1+t})\right)$$

$$-\frac{\partial u_{z_0}(x)}{\partial x}|_{x^*} \sum_{t=0}^{l-1} \delta^t \left(x_{1+t} - y_{1+t}\right) \leq \sum_{t \geq l} \delta^t M(z_0 - z_l)|x_{1+t} - y_{1+t}|$$

$$-\frac{\partial u_{z_0}(x)}{\partial x}|_{x^*} \max_{0 \leq t \leq l-1} \{|x_{1+t} - y_{1+t}|\} \leq \frac{1}{1-\delta} M(z_0 - z_l) \sup_{t \geq l} \{|x_{1+t} - y_{1+t}|\}$$

$$\left(-\frac{\partial u_{z_0}(x)}{\partial x}|_{x^*}\right) \frac{1-\delta}{M(z_0 - z_l)} \max_{0 \leq t \leq l-1} \{|x_{1+t} - y_{1+t}|\} \leq \sup_{t \geq l} \{|x_{1+t} - y_{1+t}|\}$$

Since $z_0 - z_l \leq \epsilon - d^-_{x^*+\epsilon} + d^-_{x^*} \xrightarrow[\epsilon \to 0]{} 0$, by taking ϵ small enough, we can guarantee

$$D \max_{0 \leq t \leq l-1} \{|x_{1+t} - y_{1+t}|\} \leq \sup_{t \geq l} \{|x_{1+t} - y_{1+t}|\}$$

for a fixed $D > 2$ (we can take D arbitrarily large). Take $t_0 = \arg\max_{0 \leq t \leq l-1} \{|x_{1+t} - y_{1+t}|\}$ and t_1 the smallest $t \geq l$ for which $|x_{1+t_1} - y_{1+t_1}| \geq 2|x_{1+t_0} - y_{1+t_0}|$. We can apply the same argument to obtain t_2 such that $|x_{1+t_2} - y_{1+t_2}| \geq 2|x_{1+t_1} - y_{1+t_1}|$, and so on for t_3, etc. Then, for large enough j, $|x_{1+t_j} - y_{1+t_j}| > x^{**} - x^*$, a contradiction.

This argument proves (i) for an interval $[x^*, x^* + \epsilon)$. Now let

$$\hat{x} = \inf\{\tilde{x} : s \text{ is not monotonic on } [x^*, \tilde{x}]\} \geq x^* + \epsilon.$$

Suppose WLOG that $x^{**} > m^{-1}(x^* + d^+_{x^*})$. We will now show that $\hat{x} \geq m^{-1}(x^* + d^+_{x^*}) > x^* + d^+_{x^*}$. Suppose $\hat{x} < m^{-1}(x^* + d^+_{x^*})$.

By construction, for any $\varepsilon > 0$, there must be pairs $x^\varepsilon, y^\varepsilon$ such that $x^\varepsilon < y^\varepsilon$, $s(x^\varepsilon) > s(y^\varepsilon)$ and $x^\varepsilon, y^\varepsilon \in (\hat{x} - \varepsilon, \hat{x} + \varepsilon)$. There are two cases. If there are arbitrarily small ε for which $s^k(x^\varepsilon), s^k(y^\varepsilon) \geq \hat{x}$ for all k, and moreover $\lim_{k\to\infty} s^k(x^\varepsilon) = \lim_{k\to\infty} s^k(y^\varepsilon) \geq \hat{x}$, then we obtain a contradiction by repeating our previous argument. Else we are able to pick x^ε and y^ε so that, in the addition to the above conditions, $s^k(x^\varepsilon) > s^k(y^\varepsilon)$ for all $k \geq 1$.[36] Label $s^k(x^\varepsilon) = x_k$, $s^k(y^\varepsilon) = y_k$.

A majority in $I(y_0)$ must prefer $S(y_1)$ to $S(x_1)$, and a majority in $I(x_0)$ must prefer $S(x_1)$ to $S(y_1)$. Then some $z_0 \in I(x_0) - I(y_0)$ prefers $S(x_1)$ to $S(y_1)$. A2 implies that all $z \in [z_0, x^* + d^+_{x^*}]$ strictly prefer $S(x_1)$ to $S(y_1)$. For ϵ small enough, $m(I(y_0)) < x^* + d^+_{x^*}$, so a strict majority in $I(y_0)$ prefers $S(x_1)$ to $S(y_1)$, a contradiction.

Next, we prove that the MVT must hold. Let $y \in I(x^*) \cap [x^*, x^{**})$ and suppose $m(y)$ strictly prefers $S(y')$ to $S(s(y))$, where $y' < s(y)$. Since s is increasing in $I(x^*) \cap [x^*, x^{**})$, $s^k(y') \leq s^k(s(y))$ for all k, so by A2 all voters $x < m(y)$ prefer $S(y')$ to $S(s(y))$. Some voters $x > m(y)$ close to $m(y)$ also prefer $S(y')$ by continuity. Hence $S(s(y))$ is not a Condorcet winner in $I(y)$, a contradiction. Next, suppose $s(y) < y' \leq y$. Then all voters in $[m(y), x^* + d^+_{x^*}]$ prefer $S(y')$ by A2, and some voters $x < m(y)$ prefer $S(y')$ by continuity. On the other hand, voters $x \in (x^* + d^+_{x^*}, y + d^+_y]$ prefer $S(y')$ to $S(s(y))$ because $x \geq y$ and $s^k(y') \geq s^k(s(y))$ for all k. Hence $s(y)$ is not a Condorcet winner in $I(y)$, a contradiction.

For the existence, it is enough to prove existence of MVE for the model in Section 5.[37] For this, we refine an incomplete argument given in Acemoglu et al. (2015) (Theorem B3). Briefly, for any finite policy space $X \subseteq [-1, 1]$, an MVE can be found by backward induction, and its monotonicity can be proved as above. Take a sequence of finite spaces $X_1 \subseteq X_2 \subseteq \ldots$ such that $\cup_{i\in\mathbb{N}} X_i$ is dense in $[-1, 1]$, and take an MVE

[36] If eventually $s^k(x^\varepsilon) < \hat{x}$, or $s^k(x^\varepsilon)$ and $s^k(y^\varepsilon)$ converge to different limits, the inequality $s^k(x^\varepsilon) \lessgtr s^k(y^\varepsilon)$ can only flip finitely many times as k grows, and we can look at the last time it flips.

[37] Any such MVE is also an MVE of the main model within $I(x^*) \cap [x^*, x^{**})$. The reason is that any such MVE is monotonic and satisfies the MVT everywhere, i.e., $\forall y, y'$, $m(y)$ prefers $S(s(y))$ to $S(y')$. $m(y)$ then also prefers $S(s(y))$ to all $S(y')$ in the main model if $y \in I(x^*)$. By the monotonicity, and combining A2 and A6, the sets of voters in $I(y)$ preferring $S(s(y))$ to $S(y')$ or vice versa are both intervals, so a majority prefers $S(s(y))$ iff $m(y)$ does.

$\tilde{s}_i : X_i \to X_i$ for each. Let s_i be a monotonic extension of $\tilde{s}_i$ to $[-1, 1]$. By a diagonal argument, abusing notation, find a subsequence $(s_j)_j$ such that $(s_j(x))_j$ converges at every element of $\cup_{i \in \mathbb{N}} X_i$. $(s_j(x))_j$ must in fact converge at all but countably many points, so we can find a subsequence that converges for all x. Denote the limit by $\tilde{s}$. This is the construction from Acemoglu et al. (2015). But $\tilde{s}$ **need not** be an MVE, as there is no guarantee that $\tilde{S}(\tilde{s}(y)) = \lim_{j\to\infty} S_j(s_j(y))$ if $\tilde{s}$ is not continuous.

Say $S = (x_t)_t$ is an *optimal path* for y if there is a sequence $y_j \to y$ such that $S_j(s_j(y_j)) \to S$ (i.e., $s_j^t(y_j) \xrightarrow[j\to\infty]{} x_t \ \forall t$). Denote by $\mathcal{S}(y)$ the set of y's optimal paths. Then it can be shown that the elements of $\mathcal{S}(y)$ are ordered for each y; if $S \in \mathcal{S}(y)$ and $S' \in \mathcal{S}(y')$, with $y > y'$, then $S \geq S'$; and, if $S_j \in \mathcal{S}(y_j)\ \forall j$, $y_j \to y$, and $S_j \to S$, then $S \in \mathcal{S}(y)$. In addition, for any $(x_0, x_1, \ldots) \in \mathcal{S}(y)$, $(x_1, \ldots) \in \mathcal{S}(x_0)$. Moreover, $(x_1, \ldots)$ must be the maximal element of $\mathcal{S}(x_0)$, that is, it must be $\lim_{y_j \searrow x_0} \min(\mathcal{S}(y_j))$. Indeed, $m(x_0)$ is indifferent between all elements of $\mathcal{S}(x_0)$; by A2, $m(y)$ strictly prefers the maximal one. Then, if $(x_1, \ldots)$ is not the maximal element, $m(y)$ would deviate to $x_0 + \epsilon$ for $\epsilon > 0$.

Define then s by $s(y) = \inf_{y' > y} \tilde{s}(y')$. We can show by induction on t that $S(s(y)) = \max(\mathcal{S}(y))$, and from there that s constitutes an MVE. □

Proof of Proposition 4. Let $t_0 = \min\{t : s^t(x) \leq y\}$. By Proposition 1, $s^t(x) \leq s^{t_0}(x) \leq y$ for all $t \geq t_0$. A majority of $I(x)$ must prefer $S(s(x))$ to $S(x)$. Then, by Corollary 2, $m(x)$ must have this preference:

$$\frac{u_{m(x)}(x)}{1-\delta} \leq U_{m(x)}(S(s(x))) \leq \frac{1-\delta^{t_0}}{1-\delta} u_{m(x)}(m(x)) + \frac{\delta^{t_0}}{1-\delta} \max(u_{m(x)}(y), 0)$$
$$\delta^{t_0} \leq \frac{u_{m(x)}(m(x)) - u_{m(x)}(x)}{u_{m(x)}(m(x)) - \max(u_{m(x)}(y), 0)}$$

$$t_0 \frac{1-\delta}{\delta} \geq t_0 \ln\left(\frac{1}{\delta}\right) \geq \ln\left(\frac{u_{m(x)}(m(x)) - \max(u_{m(x)}(y), 0)}{u_{m(x)}(m(x)) - u_{m(x)}(x)}\right) =: K(y). \qquad \square$$

Proof of Proposition 5. Parts (i) and (iii) follow from the arguments given in the text plus Lemma 2.

For part (ii), we first construct a sequence of approximate Q1Es as follows. For each $i = 1, 2, \ldots$, let $\epsilon(i) = \frac{1}{i}$ and take y_1, y_2 such that $x^* < y_1 < y_2 < x^* + \epsilon(i)$ and such that, moreover, $u_{m(y_2)}(y_2) < u_{m(y_2)}(y_1)$. Define $\tilde{x}_{ik} = y_1$ for all $k > 0$ and

$\tilde{x}_{i0} = y_2$. Then, for $k = -1, -2, \ldots$ define $\tilde{x}_{ik}$ such that $m(\tilde{x}_{ik})$ is indifferent between the policy $\tilde{x}_{i(k+1)}$ and the path $(\tilde{x}_{i(k+2)}, \tilde{x}_{i(k+3)}, \ldots)$. (We can show by induction that $\tilde{x}_{ik}$ is uniquely defined and strictly decreasing in k for all $k < 0$, by Corollary 2.) Let $\tilde{s}_i$ denote the associated successor function, i.e., $\tilde{s}_i(y) = \tilde{x}_{i(k+1)}$ for all $y \in [\tilde{x}_{ik}, \tilde{x}_{i(k-1)})$.

We now make some useful observations. First, $\tilde{s}_i$ satisfies all the conditions to be a Q1E for $k < 0$. Indeed, $m(\tilde{x}_{ik})$ is indifferent between $S(\tilde{x}_{i(k+1)})$ and $S(\tilde{x}_{i(k+2)})$ by construction; by A2 and Corollary 2, she prefers these policy paths to any other $S(\tilde{x}_{ik'})$. Second, it can be shown by induction that $\tilde{x}_{ik}(y_1, y_2)$ is a continuous function for all $k < 0$. Third, $\tilde{x}_{ik} \leq m^{-1}(\tilde{x}_{i(k+1)})$ for all $k < 0$; in particular, $\tilde{x}_{ik} \leq m^k(\tilde{x}_{i0}) = m^k(y_2)$. Fourth, $\tilde{x}_{ik} \xrightarrow[k \to -\infty]{} x^{**}$.[38]

Next, we argue that y_1, y_2 can be chosen so that some element of the sequence $(\tilde{x}_{ik})_k$ equals $\underline{x}$. For an arbitrary initial choice of y_1, y_2 satisfying the requirements above, let k_0 be such that $\underline{x} > \tilde{x}_{ik_0}$. Now lower (y_1, y_2) continuously towards x^* while satisfying the conditions that $x^* < y_1 < y_2 < x^* + \epsilon(i)$ and $u_{m(y_2)}(y_2) < u_{m(y_2)}(y_1)$. Then $\tilde{x}_{ik_0}(y_1, y_2) \leq m^k(y_2) \xrightarrow[y_2 \to x^*]{} x^*$, so there are intermediate values of y_1, y_2 for which $\tilde{x}_{ik_0}(y_1, y_2) = \underline{x}$. Denote $y_{i1} = y_1$, $y_{i2} = y_2$, $x_{ik} = \tilde{x}_{i(k+k_0)}(y_{i1}, y_{i2})$, $s_i = \tilde{s}_i(y_{i1}, y_{i2})$. Note that $x_{i0} = \underline{x}$ for all i.

We now construct a true Q1E s by taking the limit of a subsequence of s_i. We use a diagonal argument: $x_{i0} \to x_0 = \underline{x}$ by construction. For all i, x_{i1} is contained in $[x^*, \underline{x}]$, so we can take a convergent subsequence such that $x_{i_j 1} \to x_1$. Next, we take a subsequence such that the $x_{i_{j_l} 2}$ also converge, etc. By an abuse of notation, let x_{jk} denote the result of this argument, so that $x_{jk} \to x_k$ for all k.

The indifference conditions that made the s_i Q1Es under m_i make s a Q1E under m by continuity. To guarantee that s is a proper Q1E, we must also show that $x_k > x_{k+1}$ for all k; $x_k \xrightarrow[k \to +\infty]{} x^*$; and $x_k \xrightarrow[k \to -\infty]{} x^{**}$.

For all these claims it is enough to show that there cannot be two sequences $x_{ik(i)}$, $x_{ik'(i)}$ such that $k(i) < k'(i)$ for all i but $\lim_{i\to\infty} x_{ik(i)} = \lim_{i\to\infty} x_{ik'(i)} \in (x^*, x^{**})$. In turn, it is enough to show that this cannot happen for $k'(i) = k(i)+1$. Suppose it does, and relabel the sequences as follows: $y_{il} = x_{i(l+k(i))}$. (If necessary, take a subsequence such that $y_{il} \to y_l$ for all l.) Then we just have to show that $y_0 > y_1$. Clearly $y_0 \geq y_1$ as $y_{i0} > y_{i1}$ for all i, so suppose $y_0 = y_1$. If $y_2 > y_1$, then $m(y_0) = m(y_1)$ must be indifferent between y_1, $S(y_2)$ and y_2, which implies $y_2 < m(y_1) < y_1$. But then $m(y_{1i})$

[38] If $\tilde{x}_{ik} \xrightarrow[k \to -\infty]{} y < x^{**}$, we obtain a contradiction by the same argument as in Proposition 2.

would strictly prefer y_{i2} to $S(y_{i2})$ for high enough i, a contradiction. Hence $y_1 = y_2$, and by the same argument $y_2 = y_3 = y_4 = \ldots$.

This will lead to a contradiction by a similar argument as in Proposition 2. Let $V(y) = (1-\delta)U_{m(y)}(S(s(y))) - u_{m(y)}(y)$ as in that proof. The fact that $y_1 = y_0$ implies that $V_i(y_{i0}) \to 0$. Now take an arbitrary sequence $(g(i))_i \subseteq \mathbb{N}$, and denote $y_{ig(i)} = y_{i0} - \epsilon_i$. Then, by the argument in Proposition 2,

$$V_i(y_{i0}) \geq (1-\delta)\epsilon_i \left(-\frac{\partial}{\partial x} u_{m(y_{i0})}(\tilde{y})\right) + \delta \left(V(y_{i0}) - M\epsilon_i(E(S(y_{i1})) - E(S(y_{i(g(i)+1)})))\right)$$

$$V_i(y_{i0}) \geq \epsilon_i \left(-\frac{\partial}{\partial x} u_{m(y_{i0})}(\tilde{y}_i) - \frac{\delta}{1-\delta} M(E(S(y_{i1})) - E(S(y_{i(g(i)+1)})))\right)$$

for some $\tilde{y}_i \in (y_{ig(i)}, y_{i0})$, where $E(S(y)) = (1-\delta)\sum_{t=0}^{\infty} \delta^t s^t(y)$. Given some $0 < \epsilon' < \epsilon$ and $i \in \mathbb{N}$, we say that $g(i) \in \mathbb{N}$ is ϵ',ϵ-*valid* if $\epsilon_i \in (\epsilon', \epsilon)$ and $E(S(y_{i1})) - E(S(y_{i(g(i)+1)})) \leq \frac{1-\delta}{2\delta} \frac{-\frac{\partial}{\partial x} u_{m(y_{i0})}(y_{i0})}{M}$. Clearly, if there are $0 < \epsilon' < \epsilon$ with ϵ small enough, for which we can find valid $g(i)$ for arbitrarily high i, we obtain a contradiction, as $\liminf V_i(y_{i0}) > 0$. If not, then there must be a fixed $\epsilon > 0$ and a sequence $\epsilon'_i \to 0$ for which there are no ϵ'_i,ϵ-valid values of $g(i)$ for any $i \geq i_0$. If there are arbitrarily high values of i for which $(y_{ik})_i \cap (y_{i0} - \epsilon, y_{i0} - \epsilon'_i)$ is empty, then let $y_{ih(i)}$ be the last element to the right of this gap, i.e., $y_{i(h(i)+1)} < y_{i0} - \epsilon, y_{i0} - \epsilon'_i < y_{ih(i)}$ and relabel the sequence so that $z_{i0} = y_{ih(i)}$. Then $z_1 < z_0 = y_0 = z_{-1}$, which leads to a contradiction by our previous argument. If there are arbitrarily high values of i for which there is $g(i)$ such that $\epsilon_i \in (\epsilon'_i, \epsilon)$, but $E(S(y_{i1})) - E(S(y_{i(g(i)+1)})) > C$ for a fixed C, this implies that there are fixed C' and k_0 such that $y_{i0} - y_{i(g(i)+k_0)} > C'$, and hence $V_i(y_{i(g(i)+k)}) \geq C''$ for some $0 < k \leq k_0$. Note that k_0, C, C' and C'' are fixed even as we take ϵ to 0, which implies that $\frac{\partial V_i(y)}{\partial y}$ must become arbitrarily large and negative as $i \to \infty$, a contradiction.

Next, we show that s is a 1E in $[x^*, x^* + d^-_{x^*}]$ iff $m(x_n) < x_{n+2}$ for all n.

Note that $m(x_n) < x_{n+1}$ always holds (otherwise $m(x_n)$ would strictly prefer x_{n+1} to $S(x_{n+2})$). If $m(x_n) > x_{n+2}$, $m(x_n)$ prefers $m(x_n)$ to x_{n+2}; hence he prefers $S(m(x_n))$ to $S(x_{n+2})$, and hence to $S(x_{n+1})$. This implies that $S(x_n)$ cannot be a Condorcet winner in $I(x_n)$, as the MVT must hold in this interval by Proposition 3, and thus s is not a 1E.

Conversely, suppose that, for some $x \in [x_{n+1}, x_n)$, $I(x)$ prefers $S(y)$ to $S(x_{n+2})$ for some $y \in [x_k, x_{k-1})$. If $k \leq n+2$, this is impossible as all agents in $[x - d^-_x, m(x) + \epsilon]$

would strictly prefer $S(x_{n+2})$ to $S(y)$. Suppose then that $k \geq n+3$. By the MVT, $m(x)$ prefers $S(y)$ to $S(x_{n+2})$. Suppose $m(x) \in [x_b, x_{b-1})$; we will argue that $b = k$. If $b < k$, $m(x)$ prefers $S(x_{n+2})$ to $S(x_b)$ to $S(y)$, a contradiction. If $b > k$, $m(x)$ prefers $S(x_{n+2})$ to $S(x_k)$ to $S(y)$, a contradiction.

Next, note that, if indeed $m(x)$ prefers $S(y)$ to $S(x_{n+2})$, she then prefers $S(y)$ to $S(x_{k-1})$, and so do all agents z such that $y - d_y^- < z < m(x)$ by A2. Hence a majority in $I(x_{k-2})$ should prefer $S(y)$ to $S(x_{k-1})$. By the above argument, since $y \in [x_k, x_{k-1})$ it must be that $m(x_{k-2}) \in [x_k, x_{k-1})$, a contradiction. $\square$

Proof of Remark 1. This follows from $e^{-rt}u_\alpha(s(x,t)) = e^{-\tilde{r}\frac{rt}{\tilde{r}}}u_\alpha\left(s\left(x, \frac{\tilde{r}}{r}\frac{rt}{\tilde{r}}\right)\right)$. $\square$

Proof of Remark 2. If there are $x_1 < x_2 < x_3 \in (x-d, x+d)$ such that $f(x_1), f(x_3) < f(x_2)$, then there is a local maximum of f in $(x_1, x_3) \subseteq (x-d, x+d)$. Hence, if there is no local maximum, there must be $x^* \in (x-d, x+d)$ such that f is decreasing in $(x-d, x^*]$ and increasing in $[x^*, x+d)$. Suppose WLOG that $f(x-d) \leq f(x+d)$. By definition, $F(m(x)) - F(x-d) = \frac{F(x+d)-F(x-d)}{2}$; this implies $f(x)m'(x) = \frac{f(x+d)+f(x-d)}{2}$, given that $m(x) = x$. Since x is a stable steady state, $m'(x) < 1$, so $f(x) > \frac{f(x+d)+f(x-d)}{2} \geq f(x-d)$. Hence $x > x^*$. But then $f|_{(x-d,x)} \leq f(x) \leq f|_{(x,x+d)}$, where the first inequality is sometimes strict. Hence $F(x+d) - F(x) > F(x) - F(x-d)$, which contradicts $m(x) = x$. The other case is analogous. $\square$

Proof of Remark 3. There must be $\hat{d}$ such that f is strictly increasing in $[x^* - \hat{d}, x^*]$ and strictly decreasing in $[x^*, x^* + \hat{d}]$. Take $\overline{d} = \frac{\hat{d}}{2}$. Then, for all $d < \overline{d}$, f is strictly increasing in $[x^* - 2d, x^*]$ and strictly decreasing in $[x^*, x^* + 2d]$, so $m(x^* - d) > x^* - d$ and $m(x^* + d) < x^* + d$. Hence there is a stable steady state in $[x^* - d, x^* + d]$. $\square$

Additional proofs and robustness checks are found in the online Appendices B-E.

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