Econometrica Supplementary Material

SUPPLEMENT TO "SIEVE WALD AND QLR INFERENCE ON SEMI/NONPARAMETRIC CONDITIONAL MOMENT MODELS"

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This Supplemental Material consists of Appendices B, C, and D to the main text.

APPENDIX B: ADDITIONAL RESULTS AND PROOFS OF THE RESULTS IN THE MAIN TEXT

IN APPENDIX B, we provide the proofs of all the lemmas, theorems, and propositions stated in the main text. Additional results on consistent sieve variance estimators and bootstrap sieve t statistics are also presented.

B.1. *Proofs for Section 3 on Basic Conditions*

PROOF OF LEMMA 3.3: For *Result (1)*. Observe that $\frac{d\phi(\alpha_0)}{d\alpha}[\cdot]$ is bounded on $(\mathbf{V}, \|\cdot\|)$; and in this case equation (3.4) holds. By definitions of v_n^* and v^*, we have: $\frac{d\phi(\alpha_0)}{d\alpha}[v] = \langle v_n^*, v\rangle$ and $\frac{d\phi(\alpha_0)}{d\alpha}[v] = \langle v^*, v\rangle$ for all $v \in \overline{\mathbf{V}}_{k(n)}$. Thus

$$\langle v^* - v_n^*, v\rangle = 0 \quad \text{for all } v \in \overline{\mathbf{V}}_{k(n)} \quad \text{and} \quad \|v^*\|^2 = \|v^* - v_n^*\|^2 + \|v_n^*\|^2.$$

Since $\overline{\mathbf{V}}_{k(n)}$ is a finite dimensional Hilbert space, we have $v_n^* = \arg\min_{v\in\overline{\mathbf{V}}_{k(n)}} \|v^* - v\|$. Since $\overline{\mathbf{V}}_{k(n)}$ is dense in $(\overline{\mathbf{V}}, \|\cdot\|)$, we have $\|v^* - v_n^*\| \to 0$ and $\|v_n^*\| \to \|v^*\| < \infty$ as $k(n) \to \infty$.

For *Result (2)*. We show this part by contradiction. That is, assume that $\lim_{k(n)\to\infty} \|v_n^*\| = C^* < \infty$. Since $\frac{d\phi(\alpha_0)}{d\alpha}$ is unbounded under $\|\cdot\|$ in $\mathbf{V}$, we have: for any $M > 0$, there exists a $v_M \in \mathbf{V}$ such that $|\frac{d\phi(\alpha_0)}{d\alpha}[v_M]| > M\|v_M\|$.

Since $v_M \in \mathbf{V}$, and $\{\mathbf{V}_k\}_k$ is dense (under $\|\cdot\|_s$) in $\mathbf{V}$, there exists a sequence $(v_{n,M})_n$ such that $v_{n,M} \in \mathbf{V}_{k(n)}$ and $\lim_{n\to\infty} \|v_{n,M} - v_M\|_s = 0$. This result and the fact that $\|\cdot\| \le C\|\cdot\|_s$ for some finite $C > 0$ imply that $\lim_{n\to\infty} \|v_{n,M}\| = \|v_M\|$. Also, since $\frac{d\phi(\alpha_0)}{d\alpha}[\cdot]$ is continuous or bounded on $(\mathbf{V}, \|\cdot\|_s)$, we have

$$\lim_{n\to\infty}\left|\frac{d\phi(\alpha_0)}{d\alpha}[v_{n,M} - v_M]\right| = 0.$$

Hence, there exists a $N(M)$ such that

$$\left|\frac{d\phi(\alpha_0)}{d\alpha}[v_{n,M}]\right| \ge M\|v_{n,M}\|$$

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for all $n \geq N(M)$. Since $v_{n,M} \in \mathbf{V}_{k(n)}$, the previous inequality implies that

$$\left\|v_n^*\right\| = \sup_{v \in \overline{\mathbf{V}}_{k(n)}:\|v\|\neq 0} \frac{\left|\frac{d\phi(\alpha_0)}{d\alpha}[v]\right|}{\|v\|} \geq M$$

for all $n \geq N(M)$. Since M is arbitrary, we have $\lim_{k(n)\to\infty} \|v_n^*\| = \infty$. A contradiction. *Q.E.D.*

B.2. *Proofs for Section 4 on Sieve t (Wald) and SQLR*

LEMMA B.1: *Let* $\widehat{\alpha}_n$ *be the PSMD estimator* (2.2) *and conditions for Lemma* 3.2 *hold*. *Let Assumptions* 3.5(i) *and* 3.6(i) *hold*. *Then*:

$$\sqrt{n}\langle u_n^*, \widehat{\alpha}_n - \alpha_0\rangle = -\sqrt{n}\mathbb{Z}_n + o_{P_{Z^\infty}}(1).$$

PROOF: We note that $n^{-1}\sum_{i=1}^n \|\widehat{m}(X_i,\alpha)\|^2_{\widehat{\Sigma}^{-1}} = \widehat{Q}_n(\alpha)$. By Assumption 3.6(i), we have: for any $\epsilon_n \in \mathcal{T}_n$,

$$n^{-1}\sum_{i=1}^n \left\|\widehat{m}\big(X_i, \widehat{\alpha}_n + \epsilon_n u_n^*\big)\right\|^2_{\widehat{\Sigma}^{-1}} - n^{-1}\sum_{i=1}^n \left\|\widehat{m}(X_i, \widehat{\alpha}_n)\right\|^2_{\widehat{\Sigma}^{-1}} = 2\epsilon_n\left\{\mathbb{Z}_n + \left\langle u_n^*, \widehat{\alpha}_n - \alpha_0\right\rangle\right\} + \epsilon_n^2 B_n + o_{P_{Z^\infty}}\left(r_n^{-1}\right), \tag{B.1}$$

where $r_n^{-1} = \max\{\epsilon_n^2, \epsilon_n n^{-1/2}, s_n^{-1}\}$ with $s_n^{-1} = o(n^{-1})$, and

$$\mathbb{Z}_n = n^{-1}\sum_{i=1}^n \left(\frac{dm(X_i,\alpha_0)}{d\alpha}\left[u_n^*\right]\right)' \Sigma(X_i)^{-1}\rho(Z_i,\alpha_0).$$

By adding

$$E_n(\widehat{\alpha}_n, \epsilon_n) \equiv o\big(n^{-1}\big) + \lambda_n\left(\mathrm{Pen}\left(\widehat{h}_n + \epsilon_n \frac{v_{h,n}^*}{\left\|v_n^*\right\|_{\mathrm{sd}}}\right) - \mathrm{Pen}(\widehat{h}_n)\right)$$

to both sides of equation (B.1), we have, by the definition of the approximate minimizer $\widehat{\alpha}_n$ and the fact $\widehat{\alpha}_n + \epsilon_n u_n^* \in \mathcal{A}_{k(n)}$, that, for all $\epsilon_n \in \mathcal{T}_n$,

$$2\epsilon_n\left\{\mathbb{Z}_n + \left\langle u_n^*, \widehat{\alpha}_n - \alpha_0\right\rangle\right\} + \epsilon_n^2 B_n + E_n(\widehat{\alpha}_n, \epsilon_n) + o_{P_{Z^\infty}}\left(r_n^{-1}\right) \geq 0.$$

Or, equivalently, for any $\delta > 0$ and some $N(\delta)$,

$$P_{Z^\infty}\big(\forall \epsilon_n : \widehat{\alpha}_n + \epsilon_n u_n^* \in \mathcal{N}_{osn}, 2\epsilon_n\left\{\mathbb{Z}_n + \left\langle u_n^*, \widehat{\alpha}_n - \alpha_0\right\rangle\right\} + \epsilon_n^2 B_n + E_n(\widehat{\alpha}_n, \epsilon_n) \geq -\delta r_n^{-1}\big) \geq 1 - \delta \tag{B.2}$$

for all $n \geq N(\delta)$. In particular, this holds for $\epsilon_n \equiv \pm\{s_n^{-1/2} + o(n^{-1/2})\} = \pm o(n^{-1/2})$ since $s_n^{-1/2} = o(n^{-1/2})$. Under this choice of ϵ_n, $r_n^{-1} = \max\{s_n^{-1}, s_n^{-1/2}n^{-1/2}\}$. Moreover, Assumptions 3.2(i)(ii) and 3.4(iv) imply that $E(\widehat{\alpha}_n, \epsilon_n) = o_{P_{Z^\infty}}(n^{-1})$. Thus $\sqrt{n}\epsilon_n^{-1}E(\widehat{\alpha}_n, \epsilon_n) = o_{P_{Z^\infty}}(\sqrt{n}\epsilon_n^{-1}n^{-1}) = o_{P_{Z^\infty}}(1)$. Thus, from equation (B.2), it follows,

$$P_{Z^\infty}\big(A_{n,\delta} \geq \sqrt{n}\big\{\mathbb{Z}_n + \big\langle u_n^*, \widehat{\alpha}_n - \alpha_0\big\rangle\big\} \geq B_{n,\delta}\big) \geq 1 - \delta$$

eventually, where

$$A_{n,\delta} \equiv -0.5\sqrt{n}\epsilon_n B_n - \delta\sqrt{n}\epsilon_n^{-1}r_n^{-1} + 0.5\delta$$

and

$$B_{n,\delta} \equiv -0.5\sqrt{n}\epsilon_n B_n - 0.5\sqrt{n}\delta\epsilon_n^{-1}r_n^{-1} - 0.5\delta$$

(here the 0.5δ follows from the previous algebra regarding $\sqrt{n}\epsilon_n^{-1}E(\widehat{\alpha}_n, \epsilon_n)$). Note that $\sqrt{n}\epsilon_n = o(1)$, $B_n = O_{P_{Z^\infty}}(1)$, and $\sqrt{n}\epsilon_n^{-1}r_n^{-1} = \pm\max\{s_n^{-1/2}\sqrt{n}, 1\} \asymp \pm 1$. Thus

$$P_{Z^\infty}\big(2\delta \geq \sqrt{n}\big\{\mathbb{Z}_n + \big\langle u_n^*, \widehat{\alpha}_n - \alpha_0\big\rangle\big\} \geq -2\delta\big) \geq 1 - \delta, \quad \text{eventually.}$$

Hence we have established $\sqrt{n}\langle u_n^*, \widehat{\alpha}_n - \alpha_0\rangle = -\sqrt{n}\mathbb{Z}_n + o_{P_{Z^\infty}}(1)$. *Q.E.D.*

PROOF OF THEOREM 4.1: By Lemma B.1 and Assumption 3.6(ii), we immediately obtain: $\sqrt{n}\langle u_n^*, \widehat{\alpha}_n - \alpha_0\rangle \Rightarrow N(0, 1)$. Hence, in order to show the result, it suffices to prove that

$$\sqrt{n}\frac{\phi(\widehat{\alpha}_n) - \phi(\alpha_0)}{\|v_n^*\|_{\mathrm{sd}}} = \sqrt{n}\big\langle u_n^*, \widehat{\alpha}_n - \alpha_0\big\rangle + o_{P_{Z^\infty}}(1).$$

By the Riesz representation theorem and the orthogonality property of $\alpha_{0,n}$, it follows

$$\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n - \alpha_{0,n}] = \big\langle v_n^*, \widehat{\alpha}_n - \alpha_{0,n}\big\rangle = \big\langle v_n^*, \widehat{\alpha}_n - \alpha_0\big\rangle.$$

By Assumptions 3.1(iv) and 3.5(i), we have $\|v_n^*\|_{\mathrm{sd}} \asymp \|v_n^*\|$. This and Assumption 3.5(ii)(iii) imply

$$\begin{aligned}
&\sqrt{n}\frac{\phi(\widehat{\alpha}_n) - \phi(\alpha_0)}{\|v_n^*\|_{\mathrm{sd}}} \\
&\quad = \sqrt{n}\|v_n^*\|_{\mathrm{sd}}^{-1}\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n - \alpha_0] + o_{P_{Z^\infty}}(1) \\
&\quad = \sqrt{n}\|v_n^*\|_{\mathrm{sd}}^{-1}\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n - \alpha_{0,n}]
\end{aligned}$$

$$+\sqrt{n}\|v_n^*\|_{\mathrm{sd}}^{-1}\frac{d\phi(\alpha_0)}{d\alpha}[\alpha_{0,n}-\alpha_0]+o_{P_{Z^\infty}}(1)$$
$$=\sqrt{n}\|v_n^*\|_{\mathrm{sd}}^{-1}\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n-\alpha_{0,n}]+o_{P_{Z^\infty}}(1)$$
$$=\sqrt{n}\|v_n^*\|_{\mathrm{sd}}^{-1}\langle v_n^*,\widehat{\alpha}_n-\alpha_0\rangle+o_{P_{Z^\infty}}(1).$$

Thus

$$\sqrt{n}\frac{\phi(\widehat{\alpha}_n)-\phi(\alpha_0)}{\|v_n^*\|_{\mathrm{sd}}}=\sqrt{n}\frac{\langle v_n^*,\widehat{\alpha}_n-\alpha_0\rangle}{\|v_n^*\|_{\mathrm{sd}}}+o_{P_{Z^\infty}}(1),$$

and the claimed result now follows from Lemma B.1 and Assumption 3.6(ii).
Q.E.D.

PROOF OF LEMMA 4.1: By the definitions of $\overline{\mathbf{V}}_{k(n)}$ and the sieve Riesz representer $v_n^*\in\overline{\mathbf{V}}_{k(n)}$ of $\frac{d\phi(\alpha_0)}{d\alpha}[\cdot]$ given in (3.6), we know that $v_n^*=(v_{\theta,n}^{*\prime},v_{h,n}^*(\cdot))'=(v_{\theta,n}^{*\prime},\psi^{k(n)}(\cdot)'\beta_n^*)'\in\overline{\mathbf{V}}_{k(n)}$ solves the following optimization problem:

$$\text{(B.3)}\qquad \frac{d\phi(\alpha_0)}{d\alpha}[v_n^*]$$
$$=\|v_n^*\|^2$$
$$=\sup_{v=(v_\theta',v_h)'\in\overline{\mathbf{V}}_{k(n)},v\neq0}\frac{\left|\frac{\partial\phi(\alpha_0)}{\partial\theta'}v_\theta+\frac{\partial\phi(\alpha_0)}{\partial h}[v_h(\cdot)]\right|^2}{E\left[\left(\frac{dm(X,\alpha_0)}{d\alpha}[v]\right)'\Sigma(X)^{-1}\left(\frac{dm(X,\alpha_0)}{d\alpha}[v]\right)\right]}$$
$$=\sup_{\gamma=(v_\theta',\beta')'\in\mathbb{R}^{d_\theta+k(n)},\gamma\neq0}\frac{\gamma'F_nF_n'\gamma}{\gamma'D_n\gamma},$$

where $D_n=E[(\frac{dm(X,\alpha_0)}{d\alpha}[\overline{\psi}^{k(n)}(\cdot)'])'\Sigma(X)^{-1}(\frac{dm(X,\alpha_0)}{d\alpha}[\overline{\psi}^{k(n)}(\cdot)'])]$ is a $(d_\theta+k(n))\times(d_\theta+k(n))$ positive definite matrix such that

$$\gamma'D_n\gamma\equiv E\left[\left(\frac{dm(X,\alpha_0)}{d\alpha}[v]\right)'\Sigma(X)^{-1}\left(\frac{dm(X,\alpha_0)}{d\alpha}[v]\right)\right]$$
$$\text{for all } v=\left(v_\theta',\psi^{k(n)}(\cdot)'\beta\right)'\in\overline{\mathbf{V}}_{k(n)},$$

and $F_n\equiv(\frac{\partial\phi(\alpha_0)}{\partial\theta'},\frac{\partial\phi(\alpha_0)}{\partial h}[\psi^{k(n)}(\cdot)'])'=\frac{d\phi(\alpha_0)}{d\alpha}[\overline{\psi}^{k(n)}(\cdot)]$ is a $(d_\theta+k(n))\times1$ vector.

The sieve Riesz representation (3.6) becomes: for all $v = (v_\theta', \psi^{k(n)}(\cdot)'\beta)' \in \overline{\mathbf{V}}_{k(n)}$,

$$\frac{d\phi(\alpha_0)}{d\alpha}[v] = F_n'\gamma = \langle v_n^*, v\rangle = \gamma_n^{*\prime} D_n \gamma \quad \text{for all } \gamma = (v_\theta', \beta')' \in \mathbb{R}^{d_\theta + k(n)}. \tag{B.4}$$

It is obvious that the optimal solution of γ in (B.3) or in (B.4) has a closed form expression:

$$\gamma_n^* = (v_{\theta,n}^{*\prime}, \beta_n^{*\prime})' = D_n^- F_n.$$

The sieve Riesz representer is then given by

$$v_n^* = (v_{\theta,n}^{*\prime}, v_{h,n}^*(\cdot))' = (v_{\theta,n}^{*\prime}, \psi^{k(n)}(\cdot)'\beta_n^*)' \in \overline{\mathbf{V}}_{k(n)}.$$

Consequently, $\|v_n^*\|^2 = \gamma_n^{*\prime} D_n \gamma_n^* = F_n' D_n^- F_n$. *Q.E.D.*

Another consistent variance estimator. For $\|v_n^*\|_{\mathrm{sd}}^2 = E(S_{n,i}^* S_{n,i}^{*\prime})$ given in (3.8) and (4.3), by Lemma 4.1, it has an alternative closed form expression:

$$\begin{aligned}
\|v_n^*\|_{\mathrm{sd}}^2 &= F_n' D_n^- \Omega_n D_n^- F_n, \\
\Omega_n &\equiv E\left[\left(\frac{dm(X,\alpha_0)}{d\alpha}[\overline{\psi}^{k(n)}(\cdot)']\right)' \Sigma(X)^{-1}\Sigma_0(X)\Sigma(X)^{-1} \right. \\
&\qquad \left. \times \left(\frac{dm(X,\alpha_0)}{d\alpha}[\overline{\psi}^{k(n)}(\cdot)']\right)\right] \\
&= \mho_n.
\end{aligned}$$

Therefore, in addition to the sieve variance estimator $\|\widehat{v}_n^*\|_{n,\mathrm{sd}}$ given in (4.7), we can define another simple plug-in sieve variance estimator:

$$\begin{aligned}
\|\widehat{v}_n^*\|_{n,\mathrm{sd}}^2 &= \|\widehat{v}_n^*\|_{n,\widehat{\Sigma}^{-1}\widehat{\Sigma}_0\widehat{\Sigma}^{-1}}^2 \\
&= \frac{1}{n}\sum_{i=1}^n \left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n)}{d\alpha}[\widehat{v}_n^*]\right)' \widehat{\Sigma}_i^{-1}\widehat{\Sigma}_{0i}\widehat{\Sigma}_i^{-1}\left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n)}{d\alpha}[\widehat{v}_n^*]\right)
\end{aligned} \tag{B.5}$$

with $\widehat{\Sigma}_{0i} = \widehat{\Sigma}_0(X_i)$, where $\widehat{\Sigma}_0(x)$ is a consistent estimator of $\Sigma_0(x)$, for example, $\widehat{E}_n[\rho(Z,\widehat{\alpha}_n)\rho(Z,\widehat{\alpha}_n)'|X = x]$, where $\widehat{E}_n[\cdot|X = x]$ is some consistent estimator of a conditional mean function of X, such as a series, kernel, or local polynomial based estimator.

The sieve variance estimator given in (B.5) can also be expressed as

$$\|\widehat{v}_n^*\|_{n,\mathrm{sd}}^2 = \widehat{V}_2 \equiv \widehat{F}'_n \widehat{D}_n^- \widehat{\Omega}_n \widehat{D}_n^- \quad \text{with} \tag{B.6}$$

$$\widehat{\Omega}_n = \frac{1}{n}\sum_{i=1}^n \left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n)}{d\alpha}\left[\overline{\psi}^{k(n)}(\cdot)'\right]\right)' \widehat{\Sigma}_i^{-1}\widehat{\Sigma}_{0i}\widehat{\Sigma}_i^{-1} \times \left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n)}{d\alpha}\left[\overline{\psi}^{k(n)}(\cdot)'\right]\right).$$

ASSUMPTION B.1: (i) $\sup_{v\in\overline{\mathbf{V}}^1_{k(n)}} |\langle v,v\rangle_{n,\Sigma^{-1}\Sigma_0\Sigma^{-1}} - \langle v,v\rangle_{\Sigma^{-1}\Sigma_0\Sigma^{-1}}| = o_{P_{Z^\infty}}(1)$; *and*

(ii) $\sup_{\alpha\in\mathcal{N}_{osn}}\sup_{x\in\mathcal{X}} \|\widehat{E}_n[\rho(z,\alpha)\rho(z,\alpha)'|X=x] - E[\rho(z,\alpha)\rho(z,\alpha)'|X=x]\|_e = o_{P_{Z^\infty}}(1)$.

THEOREM B.1: *Let Assumption* 4.1(i)–(iv), *Assumption* B.1, *and assumptions for Lemma* 3.2 *hold*. *Then*: *Results* (1) *and* (2) *of Theorem* 4.2 *hold with* $\|\widehat{v}_n^*\|^2_{n,\mathrm{sd}}$ *given in* (B.5).

Monte Carlo studies indicate that both sieve variance estimators perform well and similarly in finite samples.

PROOF OF THEOREMS 4.2 AND B.1: In the proof, we use simplified notation $o_{P_{Z^\infty}}(1) = o_P(1)$. Also, Result (2) trivially follows from Result (1) and Theorem 4.1. So we only show Result (1). For *Result (1)*, by the triangle inequality, we have that

$$\left|\frac{\|\widehat{v}_n^*\|_{n,\mathrm{sd}} - \|v_n^*\|_{\mathrm{sd}}}{\|v_n^*\|_{\mathrm{sd}}}\right| \le \left|\frac{\|\widehat{v}_n^*\|_{n,\mathrm{sd}} - \|\widehat{v}_n^*\|_{\mathrm{sd}}}{\|v_n^*\|_{\mathrm{sd}}}\right| + \left|\frac{\|\widehat{v}_n^*\|_{\mathrm{sd}} - \|v_n^*\|_{\mathrm{sd}}}{\|v_n^*\|_{\mathrm{sd}}}\right| \le \left|\frac{\|\widehat{v}_n^*\|_{n,\mathrm{sd}} - \|\widehat{v}_n^*\|_{\mathrm{sd}}}{\|v_n^*\|_{\mathrm{sd}}}\right| + \frac{\|\widehat{v}_n^* - v_n^*\|_{\mathrm{sd}}}{\|v_n^*\|_{\mathrm{sd}}}.$$

This and the fact $\frac{\|\widehat{v}_n^*-v_n^*\|_{\mathrm{sd}}}{\|v_n^*\|_{\mathrm{sd}}} \asymp \frac{\|\widehat{v}_n^*-v_n^*\|}{\|v_n^*\|}$ (under Assumption 3.1(iv)) imply that Result (1) follows from

$$\frac{\|\widehat{v}_n^* - v_n^*\|}{\|v_n^*\|} = o_P(1), \tag{B.7}$$

and

$$\left|\frac{\|\widehat{v}_n^*\|_{n,\mathrm{sd}} - \|\widehat{v}_n^*\|_{\mathrm{sd}}}{\|v_n^*\|_{\mathrm{sd}}}\right| = o_P(1). \tag{B.8}$$

We will establish results (B.7) and (B.8) in Step 1 and Step 2 below.

STEP 1: Observe that result (B.7) is about the consistency of the empirical sieve Riesz representer $\widehat{v}_n^*$ in $\|\cdot\|$ norm, which is the same whether we use $\widehat{\rho}_i\widehat{\rho}_i'$ or $\widehat{\Sigma}_{0i}$ to compute the sieve variance estimators (4.7) or (B.5). By the Riesz representation theorem, we have, for all $v \in \overline{\mathbf{V}}_{k(n)}$,

$$\frac{d\phi(\widehat{\alpha}_n)}{d\alpha}[v] = \langle \widehat{v}_n^*, v\rangle_{n,\widehat{\Sigma}^{-1}} \quad \text{and} \quad \frac{d\phi(\alpha_0)}{d\alpha}[v] = \langle v_n^*, v\rangle = \langle v_n^*, v\rangle_{\Sigma^{-1}}.$$

Hence, by Assumption 4.1(i), we have

$$\begin{aligned}
o_P(1) &= \sup_{v\in\overline{\mathbf{V}}_{k(n)}} \left| \frac{\langle \widehat{v}_n^*, v\rangle_{n,\widehat{\Sigma}^{-1}} - \langle v_n^*, v\rangle}{\|v\|}\right| \\
&= \sup_{v\in\overline{\mathbf{V}}_{k(n)}} \left| \frac{\langle \widehat{v}_n^*, v\rangle_{n,\widehat{\Sigma}^{-1}} - \langle \widehat{v}_n^*, v\rangle}{\|\widehat{v}_n^*\| \times \|v\|}\|\widehat{v}_n^*\| + \frac{\langle \widehat{v}_n^*, v\rangle - \langle v_n^*, v\rangle}{\|v\|}\right| \\
&\geq \sup_{v\in\overline{\mathbf{V}}_{k(n)}} \left| \frac{\langle \widehat{v}_n^* - v_n^*, v\rangle}{\|v\|}\right| \\
&\quad - \sup_{\varpi\in\overline{\mathbf{V}}_{k(n)}:\|\varpi\|=1} \left|\langle \widehat{\varpi}_n^*, \varpi\rangle_{n,\widehat{\Sigma}^{-1}} - \langle \widehat{\varpi}_n^*, \varpi\rangle\right| \times \|\widehat{v}_n^*\|,
\end{aligned}$$

where $\varpi \equiv v/\|v\|$ and $\widehat{\varpi}_n^* \equiv \widehat{v}_n^*/\|\widehat{v}_n^*\|$. First note that

$$\begin{aligned}
&\left|\langle \widehat{\varpi}_n^*, \varpi\rangle_{n,\widehat{\Sigma}^{-1}} - \langle \widehat{\varpi}_n^*, \varpi\rangle\right| \\
&\quad \leq \left|\langle \widehat{\varpi}_n^*, \varpi\rangle_{n,\widehat{\Sigma}^{-1}} - \langle \widehat{\varpi}_n^*, \varpi\rangle_{n,\Sigma^{-1}}\right| + \left|\langle \widehat{\varpi}_n^*, \varpi\rangle_{n,\Sigma^{-1}} - \langle \widehat{\varpi}_n^*, \varpi\rangle_{\Sigma^{-1}}\right| \\
&\quad \equiv \left|T_{1n}(\varpi)\right| + \left|T_{2n}(\varpi)\right|.
\end{aligned}$$

By Assumption 4.1(ii), we have: $\sup_{\varpi\in\overline{\mathbf{V}}_{k(n)}:\|\varpi\|=1} |T_{2n}(\varpi)| = o_P(1)$. Note that

$$\begin{aligned}
T_{1n}(\varpi) = n^{-1}\sum_{i=1}^n &\left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n)}{d\alpha}[\widehat{\varpi}_n^*]\right)' \left\{\widehat{\Sigma}^{-1}(X_i) - \Sigma^{-1}(X_i)\right\} \\
&\times \left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n)}{d\alpha}[\varpi]\right).
\end{aligned}$$

By the triangle inequality, Assumptions 3.1(iv), and 4.1(ii)(iii), we obtain

$$\begin{aligned}
&\left|T_{1n}(\varpi)\right| \\
&\quad \leq \sup_{x\in\mathcal{X}} \left\|\widehat{\Sigma}^{-1}(x) - \Sigma^{-1}(x)\right\|_e
\end{aligned}$$

$$
\begin{aligned}
&\times \sqrt{n^{-1}\sum_{i=1}^{n}\left\|\frac{d\widehat{m}(X_i,\widehat{\alpha}_n)}{d\alpha}[\widehat{\varpi}_n^*]\right\|_e^2}\sqrt{n^{-1}\sum_{i=1}^{n}\left\|\frac{d\widehat{m}(X_i,\widehat{\alpha}_n)}{d\alpha}[\varpi]\right\|_e^2} \\
&\leq o_P(1)\times O_P\Big(\sqrt{\langle\widehat{\varpi}_n^*,\widehat{\varpi}_n^*\rangle_{n,\Sigma^{-1}}}\times\sqrt{\langle\varpi,\varpi\rangle_{n,\Sigma^{-1}}}\Big) \\
&= o_P(1)\times O_P(1) = o_P(1).
\end{aligned}
$$

Hence

$$
0 < \sup_{v\in\overline{\mathbf{V}}_{k(n)},v\neq 0}\left|\frac{\langle\widehat{v}_n^*-v_n^*,v\rangle}{\|v\|}\right| = o_P\big(1+\|\widehat{v}_n^*\|\big).
$$

In particular, for $v=\widehat{v}_n^*-v_n^*$, this implies

$$
\frac{\|\widehat{v}_n^*-v_n^*\|}{\|v_n^*\|} = \frac{o_P\big(1+\|\widehat{v}_n^*\|\big)}{\|v_n^*\|}.
$$

Note that $\|v_n^*\|\geq \text{const.}>0$ and $\frac{\|\widehat{v}_n^*\|}{\|v_n^*\|}\leq\frac{\|\widehat{v}_n^*-v_n^*\|}{\|v_n^*\|}+1$, and thus, the previous equation implies

$$
\frac{\|\widehat{v}_n^*-v_n^*\|}{\|v_n^*\|}\big(1-o_P(1)\big)=o_P(1) \quad \text{and} \quad \frac{\|\widehat{v}_n^*\|}{\|v_n^*\|}=O_P(1).
$$

STEP 2: We now show that result (B.8) holds for the sieve variance estimators $\|\widehat{v}_n^*\|_{n,\mathrm{sd}}^2$ defined in (4.7) and (B.5). By Assumption 3.1(iv), we have

$$
\begin{aligned}
&\left|\frac{\|\widehat{v}_n^*\|_{n,\mathrm{sd}}-\|\widehat{v}_n^*\|_{\mathrm{sd}}}{\|v_n^*\|_{\mathrm{sd}}}\right| \\
&= \left|\frac{\|\widehat{v}_n^*\|_{n,\mathrm{sd}}-\|\widehat{v}_n^*\|_{\mathrm{sd}}}{\|\widehat{v}_n^*\|_{\mathrm{sd}}}\right|\times\frac{\|\widehat{v}_n^*\|_{\mathrm{sd}}}{\|v_n^*\|_{\mathrm{sd}}}\asymp\left|\frac{\|\widehat{v}_n^*\|_{n,\mathrm{sd}}}{\|\widehat{v}_n^*\|_{\mathrm{sd}}}-1\right|\times\frac{\|\widehat{v}_n^*\|}{\|v_n^*\|} \\
&\leq\left(\frac{\|\widehat{v}_n^*\|_{n,\mathrm{sd}}}{\|\widehat{v}_n^*\|_{\mathrm{sd}}}+1\right)\left|\frac{\|\widehat{v}_n^*\|_{n,\mathrm{sd}}}{\|\widehat{v}_n^*\|_{\mathrm{sd}}}-1\right|\times\frac{\|\widehat{v}_n^*\|}{\|v_n^*\|}=\left|\frac{\|\widehat{v}_n^*\|_{n,\mathrm{sd}}^2}{\|\widehat{v}_n^*\|_{\mathrm{sd}}^2}-1\right|\times\frac{\|\widehat{v}_n^*\|}{\|v_n^*\|} \\
&= \big|\|\widehat{\varpi}_n^*\|_{n,\mathrm{sd}}^2-\|\widehat{\varpi}_n^*\|_{\mathrm{sd}}^2\big|\times\frac{\|\widehat{v}_n^*\|^2}{\|\widehat{v}_n^*\|_{\mathrm{sd}}^2}\times\frac{\|\widehat{v}_n^*\|}{\|v_n^*\|} \\
&= \big|\|\widehat{\varpi}_n^*\|_{n,\mathrm{sd}}^2-\|\widehat{\varpi}_n^*\|_{\mathrm{sd}}^2\big|\times O_P(1),
\end{aligned}
$$

where $\widehat{\varpi}_n^* \equiv \widehat{v}_n^*/\|\widehat{v}_n^*\|$, $\frac{\|\widehat{v}_n^*\|}{\|v_n^*\|} = O_P(1)$ (by Step 1), and $\frac{\|\widehat{v}_n^*\|^2}{\|\widehat{v}_n^*\|_{\mathrm{sd}}^2} = O_P(1)$ (by Assumption 3.1(iv) and i.i.d. data). Thus, it suffices to show that

$$\left| \left\| \widehat{\varpi}_n^* \right\|_{n,\mathrm{sd}}^2 - \left\| \widehat{\varpi}_n^* \right\|_{\mathrm{sd}}^2 \right| = o_P(1). \tag{B.9}$$

STEP 2A FOR THE ESTIMATOR $\|\widehat{v}_n^*\|_{n,\mathrm{sd}}^2$ DEFINED IN (4.7): We now establish the result (B.9) when the sieve variance estimator is defined in (4.7).

Let $\widehat{M}(Z_i,\alpha) = \widehat{\Sigma}_i^{-1}\rho(Z_i,\alpha)\rho(Z_i,\alpha)'\widehat{\Sigma}_i^{-1}$ and $M(z,\alpha_0) \equiv \Sigma^{-1}(x)\rho(z,\alpha_0) \times \rho(z,\alpha_0)'\Sigma^{-1}(x)$ and $M_i = M(Z_i,\alpha_0)$. Also let $\widehat{T}_i[v_n] \equiv \frac{d\widehat{m}(X_i,\widehat{\alpha}_n)}{d\alpha}[v_n]$, $T_i[v_n] \equiv \frac{dm(X_i,\alpha_0)}{d\alpha}[v_n]$, and $\Sigma(x,\alpha) \equiv E[\rho(Z,\alpha)\rho(Z,\alpha)'|x]$.

It turns out that $|\|\widehat{\varpi}_n^*\|_{n,\mathrm{sd}}^2 - \|\widehat{\varpi}_n^*\|_{\mathrm{sd}}^2|$ can be bounded above by

$$\begin{aligned}
&\sup_{v_n\in\overline{\mathbf{V}}_{k(n)}^1}\left| n^{-1}\sum_{i=1}^n \widehat{T}_i[v_n]'\widehat{M}(Z_i,\widehat{\alpha}_n)\widehat{T}_i[v_n] - n^{-1}\sum_{i=1}^n \widehat{T}_i[v_n]'M_i\widehat{T}_i[v_n]\right| \\
&\quad + \sup_{v_n\in\overline{\mathbf{V}}_{k(n)}^1}\left| n^{-1}\sum_{i=1}^n \widehat{T}_i[v_n]'M_i\widehat{T}_i[v_n] - E\big[T_i[v_n]'M_iT_i[v_n]\big]\right| \\
&\quad + \sup_{v_n\in\overline{\mathbf{V}}_{k(n)}^1}\big|E\big[T_i[v_n]'M_iT_i[v_n]\big] \\
&\quad - E\big[T_i[v_n]'\Sigma^{-1}(X_i)\Sigma(X_i,\alpha_0)\Sigma^{-1}(X_i)T_i[v_n]\big]\big| \\
&\equiv A_{1n} + A_{2n} + A_{3n}.
\end{aligned}$$

Note that $A_{3n} = 0$ by the fact that $E[M_i|X_i] = \Sigma^{-1}(X_i)\Sigma(X_i,\alpha_0)\Sigma^{-1}(X_i)$, and that $A_{2n} = o_P(1)$ by Assumption 4.1(v). Thus it remains to show that $A_{1n} = o_P(1)$. We note that

$$\begin{aligned}
A_{1n} &\le \sup_z \sup_{\alpha\in\mathcal{N}_{osn}} \big\|\widehat{M}(z,\alpha) - M(z,\alpha_0)\big\|_e \sup_{v_n\in\overline{\mathbf{V}}_n^1}\left| n^{-1}\sum_{i=1}^n \widehat{T}_i[v_n]'\widehat{T}_i[v_n]\right| \\
&\le \text{const.} \times \sup_z \sup_{\alpha\in\mathcal{N}_{osn}} \big\|\widehat{M}(z,\alpha) - M(z,\alpha_0)\big\|_e \\
&\quad \times \sup_{v_n\in\overline{\mathbf{V}}_n^1}\left| n^{-1}\sum_{i=1}^n \widehat{T}_i[v_n]'M(Z_i,\alpha_0)\widehat{T}_i[v_n]\right|,
\end{aligned}$$

where the first inequality follows from the fact that for matrices A and B, $|A'BA| \le \|A\|_e^2\|B\|_e$ and Assumption 3.1(iv). Observe that by Assumptions

4.1(iii)(iv) and 3.1(iv),

$$\begin{aligned}
&\sup_{z}\sup_{\alpha\in\mathcal{N}_{osn}}\left\|\widehat{M}(z,\alpha)-M(z,\alpha_0)\right\|_e\\
&\quad\le\sup_{z}\sup_{\alpha\in\mathcal{N}_{osn}}\left\|\widehat{\Sigma}^{-1}(x)\left\{\rho(z,\alpha)\rho(z,\alpha)'-\rho(z,\alpha_0)\rho(z,\alpha_0)'\right\}\widehat{\Sigma}^{-1}(x)\right\|_e\\
&\qquad+\sup_{z}\left\|\widehat{\Sigma}^{-1}(x)\rho(z,\alpha_0)\rho(z,\alpha_0)'\widehat{\Sigma}^{-1}(x)\right.\\
&\qquad\left.-\Sigma^{-1}(x)\rho(z,\alpha_0)\rho(z,\alpha_0)'\Sigma^{-1}(x)\right\|_e.
\end{aligned}$$

The first term in the RHS is $o_P(1)$ by Assumptions 4.1(iii)(iv) and 3.1(iv); the second term in the RHS is also of order $o_P(1)$ by Assumptions 4.1(iii) and 3.1(iv) and the fact that $\rho(Z,\alpha_0)\rho(Z,\alpha_0)'=O_P(1)$. By Assumption 4.1(v), $\sup_{v_n\in\overline{\mathbf{V}}_n^1}|n^{-1}\sum_{i=1}^n\widehat{T}_i[v_n]'M(Z_i,\alpha_0)\widehat{T}_i[v_n]|=O_P(1)$. Hence $A_{1n}=o_P(1)$ and result (B.9) holds.

STEP 2B FOR THE ESTIMATOR $\|\widehat{v}_n^*\|_{n,\mathrm{sd}}^2$ DEFINED IN (B.5): Since we already provide a detailed proof for result (B.9) in Step 2a for the case of (4.7), here we present a more succinct proof for the case of (B.5).

By the triangle inequality,

$$\begin{aligned}
&\left|\left\|\widehat{\varpi}_n^*\right\|_{n,\mathrm{sd}}^2-\left\|\widehat{\varpi}_n^*\right\|_{\mathrm{sd}}^2\right|\\
&\quad\le\left|\left\|\widehat{\varpi}_n^*\right\|_{n,\mathrm{sd}}^2-\left\|\widehat{\varpi}_n^*\right\|_{n,\Sigma^{-1}\Sigma_0\Sigma^{-1}}^2\right|+\left|\left\|\widehat{\varpi}_n^*\right\|_{n,\Sigma^{-1}\Sigma_0\Sigma^{-1}}^2-\left\|\widehat{\varpi}_n^*\right\|_{\mathrm{sd}}^2\right|\\
&\quad\equiv R_{1n}+R_{2n}.
\end{aligned}$$

By Assumptions 3.1(iv), 4.1(iii)(iv), and B.1, we have

$$\sup_{x\in\mathcal{X}}\left\|\widehat{\Sigma}^{-1}(x)\widehat{\Sigma}_0(x)\widehat{\Sigma}^{-1}(x)-\Sigma^{-1}(x)\Sigma_0(x)\Sigma^{-1}(x)\right\|_e=o_P(1),$$

where $\widehat{\Sigma}_0(x)=\widehat{E}_n[\rho(Z,\widehat{\alpha}_n)\rho(Z,\widehat{\alpha}_n)'|x]$. Therefore, by Assumptions 3.1(iv) and 4.1(ii) and similar algebra to the one used to bound $T_{1n}(\varpi)$, we have

$$R_{1n}\le o_P(1)\times n^{-1}\sum_{i=1}^n\left\|\frac{d\widehat{m}(X_i,\widehat{\alpha}_n)}{d\alpha}\left[\widehat{\varpi}_n^*\right]\right\|_e^2=o_P(1)\times O_P(1)=o_P(1).$$

Also by Assumption B.1, $R_{2n}=o_P(1)$. Thus result (B.9) holds. *Q.E.D.*

Before we prove Theorem 4.3, we introduce some notation that will simplify the presentation of the proofs. For any $\overline{\phi}\in\mathbb{R}$, let $\mathcal{A}(\overline{\phi})\equiv\{\alpha\in\mathcal{A}:\phi(\alpha)=\overline{\phi}\}$, and $\mathcal{A}_{k(n)}(\overline{\phi})\equiv\mathcal{A}(\overline{\phi})\cap\mathcal{A}_{k(n)}$. In particular, let $\mathcal{A}^0\equiv\mathcal{A}(\phi(\alpha_0))$ and $\mathcal{A}_{k(n)}^0\equiv\mathcal{A}_{k(n)}(\phi(\alpha_0))$.

Also, we need to show that for any deviation of α of the type $\alpha + tu_n^*$, there exists a t such that $\phi(\alpha + tu_n^*)$ is "close" to $\phi(\alpha_0)$. Formally, the following lemma holds.

LEMMA B.2: *Let Assumption* 3.5 *hold*. (1) *For any* $n \in \{1, 2, \dots\}$, *any* $r \in \{r : |r| \leq 2M_n\|v_n^*\|\delta_n\}$, *and any* $\alpha \in \mathcal{N}_{osn}$, *there exists a* $t \in \mathcal{T}_n$ *such that* $\phi(\alpha + tu_n^*) - \phi(\alpha_0) = r$ *and* $\alpha + tu_n^* \in \mathcal{A}_{k(n)}$. (2) *For any* $r \in \{r : |r| \leq \|v_n^*\|\tau_n\}$ *and any* $\alpha \in \{\alpha \in \mathcal{A}_{k(n)} : \|\alpha - \alpha_0\| \leq \tau_n\}$ *with some positive sequence* $(\tau_n)_n$ *such that* $\tau_n = O(\delta_n)$, *the* t *in Part* (1) *also satisfies* $|t| \leq \max\{C\tau_n, o(n^{-1/2})\}$ *for some constant* $C > 0$.

PROOF: For Part (1), we first show that there exists a $t \in \mathcal{T}_n$ such that $\phi(\alpha + tu_n^*) - \phi(\alpha_0) = r$. By Assumption 3.5, there exists a $(F_n)_n$ such that $F_n > 0$ and $F_n = o(n^{-1/2}\|v_n^*\|)$ and, for any $\alpha \in \mathcal{N}_{osn}$ and $t \in \mathcal{T}_n$,

$$\left|\phi\left(\alpha + tu_n^*\right) - \phi(\alpha_0) - \left\langle v_n^*, \alpha - \alpha_0\right\rangle - t\frac{\left\|v_n^*\right\|^2}{\left\|v_n^*\right\|_{\mathrm{sd}}}\right| \leq F_n. \tag{B.10}$$

(Note that by Assumption 3.5, F_n does not depend on α nor t.)

For any $r \in \{|r| \leq 2M_n\|v_n^*\|\delta_n\}$, we define $(t_l)_{l=1,2}$ as

$$t_l\left\|u_n^*\right\|^2 = -\left\langle u_n^*, \alpha - \alpha_0\right\rangle + a_{l,n}F_n\left\|v_n^*\right\|_{\mathrm{sd}}^{-1} + r\left\|v_n^*\right\|_{\mathrm{sd}}^{-1},$$

where $a_l = (-1)^l 2$. Note that, by Assumption 3.5(i) (the second part), $\|u_n^*\|^{-2} \leq c^{-2}$, and thus

$$|t_l| \leq c^{-2}\left(\left\|u_n^*\right\| \times \|\alpha - \alpha_0\| + 2|F_n| \times \left\|v_n^*\right\|_{\mathrm{sd}}^{-1} + |r| \times \left\|v_n^*\right\|_{\mathrm{sd}}^{-1}\right).$$

Without loss of generality, we can re-normalize M_n so that $c^{-2}C < M_n$ and $C \geq 1$. Hence,

$$\begin{aligned}
|t_l| &\leq c^{-2}\left(\left\|u_n^*\right\| \times \|\alpha - \alpha_0\| + 2|F_n| \times \left\|v_n^*\right\|_{\mathrm{sd}}^{-1} + |r| \times \left\|v_n^*\right\|_{\mathrm{sd}}^{-1}\right) \\
&= c^{-2}\left(\left\|u_n^*\right\| \times \|\alpha - \alpha_0\| + 2|F_n| \times \left\|v_n^*\right\|_{\mathrm{sd}}^{-1} + |r| \times \left\|v_n^*\right\|^{-1}\left\|u_n^*\right\|\right) \\
&\leq c^{-2}C\left(\left\|u_n^*\right\| \times \|\alpha - \alpha_0\| + 2|F_n| \times \left\|v_n^*\right\|_{\mathrm{sd}}^{-1} + |r| \times \left\|v_n^*\right\|^{-1}\right) \\
&\leq 4M_n^2\delta_n,
\end{aligned}$$

where the third inequality follows from Assumption 3.5(i) (the second part), and the last inequality follows from the facts that $\alpha \in \mathcal{N}_{osn}$, $c^{-2}C2|F_n| \times \|v_n^*\|_{\mathrm{sd}}^{-1} = o(n^{-1/2}) \leq M_n^2\delta_n$, $r \in \{|r| \leq 2M_n\|v_n^*\|\delta_n\}$. Thus, t_l is a valid choice in the sense that $t_l \in \mathcal{T}_n$ for $l = 1, 2$.

Thus, this result and equation (B.10) imply

$$\begin{aligned}\phi(\alpha+t_1u_n^*)-\phi(\alpha_0) &\le \langle v_n^*, \alpha-\alpha_0\rangle + t_1\frac{\|v_n^*\|^2}{\|v_n^*\|_{\mathrm{sd}}}+F_n \\ &= \|v_n^*\|_{\mathrm{sd}}\big(\langle u_n^*, \alpha-\alpha_0\rangle + t_1\|u_n^*\|^2+F_n\|v_n^*\|_{\mathrm{sd}}^{-1}\big) \\ &= r-F_n<r.\end{aligned}$$

Hence, $\phi(\alpha+t_1u_n^*)-\phi(\alpha_0)<r$. Similarly,

$$\begin{aligned}\phi(\alpha+t_2u_n^*)-\phi(\alpha_0) &\ge \langle v_n^*, \alpha-\alpha_0\rangle + t_2\frac{\|v_n^*\|^2}{\|v_n^*\|_{\mathrm{sd}}}-F_n \\ &= \|v_n^*\|_{\mathrm{sd}}\big(\langle u_n^*, \alpha-\alpha_0\rangle + t_2\|u_n^*\|^2-F_n\|v_n^*\|_{\mathrm{sd}}^{-1}\big) \\ &= r+F_n>r\end{aligned}$$

and thus $\phi(\alpha+t_2u_n^*)-\phi(\alpha_0)>r$. Since $t\mapsto\phi(\alpha+tu_n^*)$ is continuous, there exists a $t\in[t_1,t_2]$ such that $\phi(\alpha+tu_n^*)-\phi(\alpha_0)=r$. Clearly, $t\in\mathcal{T}_n$.

The fact that $\alpha(t)\equiv\alpha+tu_n^*\in\mathcal{A}_{k(n)}$ for $\alpha\in\mathcal{N}_{osn}$ and $t\in\mathcal{T}_n$ follows from the fact that the sieve space $\mathcal{A}_{k(n)}$ is assumed to be convex with non-empty interior. Part (2) can be proved in the same way as that for Part (1). *Q.E.D.*

PROOF OF THEOREM 4.3: Result (2) directly follows from Result (1) with $\Sigma=\Sigma_0$ and $\|u_n^*\|=1$. The proof of Result (1) consists of several steps.

STEP 1: For any $t_n\in\mathcal{T}_n$ wpa1, by Assumption 3.6 and Lemma B.1, we have

$$\begin{aligned}&0.5\big(\widehat{Q}_n(\widehat{\alpha}_n(-t_n))-\widehat{Q}_n(\widehat{\alpha}_n)\big) \\ &\quad=-t_n\{\mathbb{Z}_n+\langle u_n^*,\widehat{\alpha}_n-\alpha_0\rangle\}+\frac{B_n}{2}t_n^2+o_{P_{Z^\infty}}(r_n^{-1}) \\ &\quad=\frac{B_n}{2}t_n^2+o_{P_{Z^\infty}}(r_n^{-1}),\end{aligned}\tag{B.11}$$

where $r_n^{-1}=\max\{t_n^2, t_nn^{-1/2}, s_n^{-1}\}$ and $s_n^{-1}=o(n^{-1})$.

And under the null hypothesis, $\widehat{\alpha}_n^R\in\mathcal{N}_{osn}\cap\mathcal{A}_{k(n)}^0$ wpa1,

$$\begin{aligned}&0.5\big(\widehat{Q}_n(\widehat{\alpha}_n^R(t_n))-\widehat{Q}_n(\widehat{\alpha}_n^R)\big) \\ &\quad=t_n\{\mathbb{Z}_n+\langle u_n^*,\widehat{\alpha}_n^R-\alpha_0\rangle\}+\frac{B_n}{2}t_n^2+o_{P_{Z^\infty}}(r_n^{-1}) \\ &\quad=t_n\mathbb{Z}_n+\frac{B_n}{2}t_n^2+o_{P_{Z^\infty}}(r_n^{-1}),\end{aligned}\tag{B.12}$$

where the last line follows from the fact that $t_n\langle u_n^*, \widehat{\alpha}_n^R - \alpha_0\rangle = o_{P_{Z^\infty}}(r_n^{-1})$. To show this, note that under the null hypothesis, $\widehat{\alpha}_n^R \in \mathcal{N}_{osn} \cap \mathcal{A}_{k(n)}^0$ wpa1. This and Assumption 3.5(ii) imply that

$$\left|\underbrace{\phi(\widehat{\alpha}_n^R) - \phi(\alpha_0)}_{=0} - \frac{d\phi(\alpha_0)}{d\alpha}\left[\widehat{\alpha}_n^R - \alpha_0\right]\right| = o_{P_{Z^\infty}}\left(n^{-1/2}\|v_n^*\|\right).$$

Thus

$$P_{Z^\infty}\left(\frac{\sqrt{n}}{\|v_n^*\|}\left|\frac{d\phi(\alpha_0)}{d\alpha}\left[\widehat{\alpha}_n^R - \alpha_0\right]\right| < \delta\right) \geq 1 - \delta$$

eventually. By similar calculations to those in the proof of Theorem 4.1, we have

$$P_{Z^\infty}\left(\sqrt{n}\left|\langle u_n^*, \widehat{\alpha}_n^R - \alpha_0\rangle\right| < \delta\right) \geq 1 - \delta, \quad \text{eventually.}$$

Hence, $\langle u_n^*, \widehat{\alpha}_n^R - \alpha_0\rangle = o_{P_{Z^\infty}}(n^{-1/2})$, and thus $t_n\langle u_n^*, \widehat{\alpha}_n^R - \alpha_0\rangle = o_{P_{Z^\infty}}(n^{-1/2}t_n) = o_{P_{Z^\infty}}(r_n^{-1})$.

STEP 2: We choose $t_n = -\mathbb{Z}_n B_n^{-1}$. Note that under Assumption 3.6, $t_n \in \mathcal{T}_n$ wpa1. By the definition of $\widehat{\alpha}_n$, we have, under the null hypothesis,

$$\begin{aligned}
&0.5\left(\widehat{Q}_n(\widehat{\alpha}_n^R) - \widehat{Q}_n(\widehat{\alpha}_n)\right)\\
&\quad \geq 0.5\left(\widehat{Q}_n(\widehat{\alpha}_n^R) - \widehat{Q}_n\left(\widehat{\alpha}_n^R(t_n)\right)\right) - o_{P_{Z^\infty}}\left(n^{-1}\right)\\
&\quad = \frac{1}{2}\mathbb{Z}_n^2 B_n^{-1} - o_{P_{Z^\infty}}\left(\max\left\{B_n^{-2}\mathbb{Z}_n^2, -B_n^{-1}\mathbb{Z}_n n^{-1/2}, s_n^{-1}\right\}\right) - o_{P_{Z^\infty}}\left(n^{-1}\right)\\
&\quad = \frac{1}{2}\mathbb{Z}_n^2 B_n^{-1} + o_{P_{Z^\infty}}\left(n^{-1}\right),
\end{aligned}$$

where the first inequality follows from the fact that, since $t_n \in \mathcal{T}_n$ and $\widehat{\alpha}_n^R \in \mathcal{N}_{osn}$ wpa1, then $\widehat{\alpha}_n^R(t_n) \in \mathcal{A}_{k(n)}$ wpa1; and the second line follows from equation (B.12) with $t_n = -\mathbb{Z}_n B_n^{-1}$.

STEP 3: We choose $t_n^* \in \mathcal{T}_n$ wpa1 such that (a) $\phi(\widehat{\alpha}_n(t_n^*)) = \phi(\alpha_0)$, $\widehat{\alpha}_n(t_n^*) \in \mathcal{A}_{k(n)}$, and (b) $t_n^* = \mathbb{Z}_n \frac{\|v_n^*\|_{\text{sd}}^2}{\|v_n^*\|^2} + o_{P_{Z^\infty}}(n^{-1/2}) = O_{P_{Z^\infty}}(n^{-1/2})$.

Suppose such a t_n^* exists; then $[r_n(t_n^*)]^{-1} = \max\{(t_n^*)^2, t_n^* n^{-1/2}, o(n^{-1})\} = O_{P_{Z^\infty}}(n^{-1})$. By the definition of $\widehat{\alpha}_n^R$, we have, under the null hypothesis,

$$\begin{aligned}
&0.5\left(\widehat{Q}_n(\widehat{\alpha}_n^R) - \widehat{Q}_n(\widehat{\alpha}_n)\right)\\
&\quad \leq 0.5\left(\widehat{Q}_n\left(\widehat{\alpha}_n(t_n^*)\right) - \widehat{Q}_n(\widehat{\alpha}_n)\right) + o_{P_{Z^\infty}}\left(n^{-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= t_n^*\{\mathbb{Z}_n + \langle u_n^*, \widehat{\alpha}_n - \alpha_0\rangle\} + \frac{B_n}{2}(t_n^*)^2 + o_{P_{Z^\infty}}(n^{-1}) \\
&= \frac{B_n}{2}\left(\mathbb{Z}_n \frac{\|v_n^*\|_{\mathrm{sd}}^2}{\|v_n^*\|^2} + o_{P_{Z^\infty}}(n^{-1/2})\right)^2 + o_{P_{Z^\infty}}(n^{-1}) \\
&= \frac{1}{2}\mathbb{Z}_n^2 B_n^{-1} + o_{P_{Z^\infty}}(n^{-1}) = \frac{1}{2}\mathbb{Z}_n^2 \frac{\|v_n^*\|_{\mathrm{sd}}^2}{\|v_n^*\|^2} + o_{P_{Z^\infty}}(n^{-1}),
\end{aligned}
$$

where the second line follows from Assumption 3.6(i) and the fact that t_n^* satisfying (b), $[r_n(t_n^*)]^{-1} = O_{P_{Z^\infty}}(n^{-1})$; the third line follows from equation (B.11) and the fact that t_n^* satisfying (b); and the last line follows from Assumptions 3.5(i) and 3.6(ii), $|B_n - \|u_n^*\|^2| = o_{P_{Z^\infty}}(1)$, and $u_n^* = v_n^*/\|v_n^*\|_{\mathrm{sd}}$.

We now show that there is a $t_n^* \in \mathcal{T}_n$ wpa1 such that (a) and (b) hold. Denote $r \equiv \phi(\widehat{\alpha}_n) - \phi(\alpha_0)$. Since $\widehat{\alpha}_n \in \mathcal{N}_{osn}$ wpa1 and $\phi(\widehat{\alpha}_n) - \phi(\alpha_0) = O_{P_{Z^\infty}}(\|v_n^*\|/\sqrt{n})$ (see the proof of Theorem 4.1), we have $|r| \leq 2M_n\|v_n^*\|\delta_n$. Thus, by Lemma B.2, there is a $t_n^* \in \mathcal{T}_n$ wpa1 such that $\widehat{\alpha}_n(t_n^*) = \widehat{\alpha}_n + t_n^* u_n^* \in \mathcal{A}_{k(n)}$ and $\phi(\widehat{\alpha}_n(t_n^*)) = \phi(\alpha_0)$, so (a) holds. Moreover, by Assumption 3.5(ii), such a choice of t_n^* also satisfies

$$
\Bigg|\underbrace{\phi(\widehat{\alpha}_n(t_n^*)) - \phi(\alpha_0)}_{=0} - \frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n - \alpha_0 + t_n^* u_n^*]\Bigg| = o_{P_{Z^\infty}}(\|v_n^*\|/\sqrt{n}).
$$

By Assumption 3.5(i) and the definition of $u_n^* = v_n^*/\|v_n^*\|_{\mathrm{sd}}$, we have: $\frac{d\phi(\alpha_0)}{d\alpha}[t_n^* u_n^*] = t_n^* \frac{\|v_n^*\|^2}{\|v_n^*\|_{\mathrm{sd}}}$. Thus

$$
P_{Z^\infty}\left(\frac{\sqrt{n}}{\|v_n^*\|}\left|\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n - \alpha_0] + t_n^* \frac{\|v_n^*\|^2}{\|v_n^*\|_{\mathrm{sd}}}\right| < \delta\right) \geq 1 - \delta
$$

eventually. By similar algebra to that in the proof of Theorem 4.1, it follows that the LHS of the equation above is majorized by

$$
\begin{aligned}
& P_{Z^\infty}\left(\frac{\sqrt{n}}{\|v_n^*\|}\left|\langle v_n^*, \widehat{\alpha}_n - \alpha_0\rangle + t_n^* \frac{\|v_n^*\|^2}{\|v_n^*\|_{\mathrm{sd}}}\right| < \delta\right) + \delta \\
&= P_{Z^\infty}\left(\frac{\sqrt{n}}{\|v_n^*\|}\left|-\mathbb{Z}_n\|v_n^*\|_{\mathrm{sd}} + t_n^* \frac{\|v_n^*\|^2}{\|v_n^*\|_{\mathrm{sd}}}\right| < \delta\right) + \delta \\
&= P_{Z^\infty}\left(\sqrt{n}\frac{\|v_n^*\|_{\mathrm{sd}}}{\|v_n^*\|}\left|-\mathbb{Z}_n + t_n^* \frac{\|v_n^*\|^2}{\|v_n^*\|_{\mathrm{sd}}^2}\right| < \delta\right) + \delta,
\end{aligned}
$$

where the second line follows from the proof of Lemma B.1. Since $\frac{\|v_n^*\|_{sd}}{\|v_n^*\|} \asymp$ const. (by Assumption 3.5(i)), we obtain

$$P_{Z^\infty}\left(\sqrt{n}\left|t_n^* - \mathbb{Z}_n \frac{\|v_n^*\|_{sd}^2}{\|v_n^*\|^2}\right| < \delta\right) \geq 1-\delta, \quad \text{eventually.}$$

Since $\sqrt{n}\mathbb{Z}_n = O_{P_{Z^\infty}}(1)$ (Assumption 3.6(ii)), we have $t_n^* = O_{P_{Z^\infty}}(n^{-1/2})$, and in fact, $\sqrt{n}t_n^* = \sqrt{n}\mathbb{Z}_n \frac{\|v_n^*\|_{sd}^2}{\|v_n^*\|^2} + o_{P_{Z^\infty}}(1)$ and hence (b) holds. *Q.E.D.*

Let $\mathcal{A}^R \equiv \{\alpha \in \mathcal{A} : \phi(\alpha) = \phi_0\}$ be the restricted parameter space. Then $\alpha_0 \in \mathcal{A}^R$ iff the null hypothesis $H_0 : \phi(\alpha_0) = \phi_0$ holds. Also, $\mathcal{A}_{k(n)}^R \equiv \{\alpha \in \mathcal{A}_{k(n)} : \phi(\alpha) = \phi_0\}$ is a sieve space for $\mathcal{A}^R$. Let $\{\overline{\alpha}_{0,n} \in \mathcal{A}_{k(n)}^R\}$ be a sequence such that $\|\overline{\alpha}_{0,n} - \alpha_0\|_s \leq \inf_{\alpha \in \mathcal{A}_{k(n)}^R} \|\alpha - \alpha_0\|_s + o(n^{-1})$.[1]

ASSUMPTION B.2: (i) $|\text{Pen}(\overline{h}_{0,n}) - \text{Pen}(h_0)| = O(1)$ *and* $\text{Pen}(h_0) < \infty$; (ii) $\widehat{Q}_n(\overline{\alpha}_{0,n}) \leq c_0 Q(\overline{\alpha}_{0,n}) + o_{P_{Z^\infty}}(n^{-1})$.

This assumption on $\overline{\alpha}_{0,n} \in \mathcal{A}_{k(n)}^R$ is the same as Assumptions 3.2(ii) and 3.3(i) imposed on $\Pi_n\alpha_0 \in \mathcal{A}_{k(n)}$, and can be verified in the same way provided that $\alpha_0 \in \mathcal{A}^R$.

PROPOSITION B.1: *Let* $\widehat{\alpha}_n^R \in \mathcal{A}_{k(n)}^R$ *be the restricted PSMD estimator* (4.10) *and* $\alpha_0 \in \mathcal{A}^R$. *Let Assumptions* 3.1, 3.2(iii), 3.3(ii), B.2, *and* $Q(\overline{\alpha}_{0,n}) + o(n^{-1}) = O(\lambda_n) = o(1)$ *hold. Then*:

(1) $\text{Pen}(\widehat{h}_n^R) = O_{P_{Z^\infty}}(1)$ *and* $\|\widehat{\alpha}_n^R - \alpha_0\|_s = o_{P_{Z^\infty}}(1)$.

(2) *Further, let* $Q(\overline{\alpha}_{0,n}) \asymp Q(\Pi_n\alpha_0)$ *and Assumptions* 3.2(ii), 3.3(i), *and* 3.4(i)(ii)(iii) *hold. Then*: $\|\widehat{\alpha}_n^R - \alpha_0\| = O_{P_{Z^\infty}}(\delta_n)$ *and* $\|\widehat{\alpha}_n^R - \alpha_0\|_s = O_{P_{Z^\infty}}(\|\alpha_0 - \Pi_n\alpha_0\|_s + \tau_n\delta_n)$.

PROOF: The proof is very similar to those for Theorem 3.2 and Remark 4.1 in Chen and Pouzo (2012a) by recognizing that $\mathcal{A}_{k(n)}^R$ is a sieve for $\alpha_0 \in \mathcal{A}^R$.

For *Result (1)*, we first want to show that $\widehat{\alpha}_n^R \in \mathcal{A}_{k(n)}^R \cap \{\text{Pen}(h) \leq M\}$ for some $M > 0$ wpa1-P_{Z^∞}. By definitions of $\widehat{\alpha}_n^R$ and $\overline{\alpha}_{0,n}$, Assumption B.2(i)(ii) and the condition that $Q(\overline{\alpha}_{0,n}) + o(n^{-1}) = O(\lambda_n)$, we have

$$\begin{aligned}\text{Pen}(\widehat{h}_n^R) &\leq \frac{\widehat{Q}_n(\overline{\alpha}_{0,n})}{\lambda_n} + \text{Pen}(\overline{h}_{0,n}) + \frac{o(n^{-1})}{\lambda_n} \\ &\leq \frac{Q(\overline{\alpha}_{0,n}) + o(n^{-1})}{\lambda_n} + O_{P_{Z^\infty}}(1) = O_{P_{Z^\infty}}(1).\end{aligned}$$

[1]Sufficient conditions for $\overline{\alpha}_{0,n} \in \mathcal{A}_{k(n)}^R$ to solve $\inf_{\alpha \in \mathcal{A}_{k(n)}^R} \|\alpha - \alpha_0\|_s$ under the null include either (a) $\mathcal{A}_{k(n)}$ is compact (in $\|\cdot\|_s$) and ϕ is continuous (in $\|\cdot\|_s$), or (b) $\mathcal{A}_{k(n)}$ is convex and ϕ is linear.

Therefore, for any $\epsilon > 0$, $\Pr(\mathrm{Pen}(\widehat{h}_n^R) \geq M) < \epsilon$ for some M, eventually.

We now show that $\Pr(\|\widehat{\alpha}_n^R - \alpha_0\|_s \geq \epsilon) = o(1)$ for any $\epsilon > 0$. Let $\mathcal{A}_{k(n)}^{R,M} \equiv \mathcal{A}_{k(n)}^{R} \cap \{\mathrm{Pen}(h) \leq M\}$ and $\mathcal{A}^{R,M} \equiv \mathcal{A}^{R} \cap \{\mathrm{Pen}(h) \leq M\}$. These sets are compact under $\|\cdot\|_s$ (by Assumption 3.2(iii) and the $\|\cdot\|_s$-continuity of ϕ). Assumptions 3.1(i)(iv) and B.2(i) imply that $\alpha_0 \in \mathcal{A}^{R,M}$ and $\overline{\alpha}_{0,n} \in \mathcal{A}_{k(n)}^{R,M}$. Under Assumption 3.1(ii), $\mathrm{cl}(\bigcup_k \mathcal{A}_k) \supseteq \mathcal{A}$ and thus $\mathrm{cl}(\bigcup_k \mathcal{A}_k^{R,M}) \supseteq \mathcal{A}^{R,M}$. Therefore $\|\overline{\alpha}_{0,n} - \alpha_0\|_s = o(1)$ by the definition of $\overline{\alpha}_{0,n}$ and the fact that $\mathcal{A}_{k(n)}^{R,M}$ is dense in $\mathcal{A}^{R,M}$.

By standard calculations, it follows that, for any $\epsilon > 0$,

$$\begin{aligned}
&\Pr\big(\big\|\widehat{\alpha}_n^R - \alpha_0\big\|_s \geq \epsilon\big)\\
&\quad \leq \Pr\Big(\inf_{\mathcal{A}_{k(n)}^{R,M}:\|\alpha-\alpha_0\|_s\geq\epsilon}\big\{\widehat{Q}_n(\alpha) + \lambda_n \mathrm{Pen}(h)\big\}\\
&\qquad \leq \widehat{Q}_n(\overline{\alpha}_{0,n}) + \lambda_n \mathrm{Pen}(\overline{h}_{0,n}) + o_P\big(n^{-1}\big)\Big) + 0.5\epsilon.
\end{aligned}$$

Moreover (up to omitted constants)

$$\begin{aligned}
&\Pr\big(\big\|\widehat{\alpha}_n^R - \alpha_0\big\|_s \geq \epsilon\big)\\
&\quad \leq \Pr\Big(\inf_{\mathcal{A}_{k(n)}^{R,M}:\|\alpha-\alpha_0\|_s\geq\epsilon}\big\{Q(\alpha) + \lambda_n \mathrm{Pen}(h)\big\}\\
&\qquad \leq Q(\overline{\alpha}_{0,n}) + \lambda_n \mathrm{Pen}(\overline{h}_{0,n}) + O_P\big(\overline{\delta}_{m,n}^2\big) + o_P\big(n^{-1}\big)\Big) + \epsilon\\
&\quad \leq \Pr\Big(\inf_{\mathcal{A}^{R,M}:\|\alpha-\alpha_0\|_s\geq\epsilon}\big\{Q(\alpha) + \lambda_n \mathrm{Pen}(h)\big\}\\
&\qquad \leq Q(\overline{\alpha}_{0,n}) + \lambda_n \mathrm{Pen}(\overline{h}_{0,n}) + O_P\big(\overline{\delta}_{m,n}^2\big) + o_P\big(n^{-1}\big)\Big) + \epsilon,
\end{aligned}$$

where the first line follows by Assumptions 3.3(ii) and B.2 and the second by $\mathcal{A}_{k(n)}^{R,M} \subseteq \mathcal{A}^{R,M}$. Since $\mathcal{A}^{R,M}$ is compact under $\|\cdot\|_s$, $\alpha_0 \in \mathcal{A}^{R,M}$ is unique, and Q is continuous (Assumption 3.1), then $\inf_{\mathcal{A}^{R,M}:\|\alpha-\alpha_0\|_s\geq\epsilon}\{Q(\alpha) + \lambda_n \mathrm{Pen}(h)\} \geq c(\epsilon) > 0$; however, the term $Q(\overline{\alpha}_{0,n}) + \lambda_n \mathrm{Pen}(\overline{h}_{0,n}) + O_P(\overline{\delta}_{m,n}^2) + o_P(n^{-1}) = o_P(1)$ and thus the desired result follows.

For *Result (2)*, we now show that $\|\widehat{\alpha}_n^R - \alpha_0\| = O_{P_{Z^\infty}}(\kappa_n)$ where $\kappa_n^2 \equiv \max\{\delta_n^2, \|\overline{\alpha}_{0,n} - \alpha_0\|^2, \lambda_n, o(n^{-1})\}$. Let $\mathcal{A}_{osn}^R = \{\alpha \in \mathcal{A}_{osn} : \phi(\alpha) = \phi(\alpha_0)\}$ and $\mathcal{A}_{os}^R = \{\alpha \in \mathcal{A}_{os} : \phi(\alpha) = \phi(\alpha_0)\}$. Result (1) implies that $\widehat{\alpha}_n^R \in \mathcal{A}_{osn}^R$ wpa1. To show Result (2), we employ analogous arguments to those for Result (1) and obtain that, for all large $K > 0$,

$$\begin{aligned}
&\Pr\big(\|\widehat{\alpha}_n^R - \alpha_0\| \geq K\kappa_n\big) \\
&\quad \leq \Pr\Big(\inf_{\mathcal{A}_{osn}^R:\|\alpha-\alpha_0\|\geq K\kappa_n} Q(\alpha) + \lambda_n \operatorname{Pen}(h) \\
&\qquad \leq Q(\overline{\alpha}_{0,n}) + \lambda_n \operatorname{Pen}(\overline{h}_{0,n}) + O_P\big(\delta_n^2\big) + o_P\big(n^{-1}\big)\Big) + \epsilon \\
&\quad \leq \Pr\Big(\inf_{\mathcal{A}_{os}^R:\|\alpha-\alpha_0\|\geq K\kappa_n} \|\alpha - \alpha_0\|^2 \\
&\qquad \leq \text{const.}\,\{\|\overline{\alpha}_{0,n} - \alpha_0\|^2 + \lambda_n \operatorname{Pen}(\overline{h}_{0,n}) + O_P\big(\delta_n^2\big) + o_P\big(n^{-1}\big)\Big) + \epsilon \\
&\quad \leq \Pr\big(K^2\kappa_n^2 \leq \text{const.}\,\|\overline{\alpha}_{0,n} - \alpha_0\|^2 + O(\lambda_n) + O_P\big(\delta_n^2\big) + o_P\big(n^{-1}\big)\big) + \epsilon,
\end{aligned}$$

where the first inequality is due to Assumption B.2(ii) and the assumption that $\widehat{Q}_n(\alpha) \geq cQ(\alpha) - O_{P_{Z^\infty}}(\delta_n^2)$ uniformly over $\mathcal{A}_{osn}$; the second inequality is due to Assumption 3.4. By our choice of κ_n the first term in the RHS is zero for large K. So the desired result follows. The fact that κ_n coincides with δ_n follows from the fact that $\|\overline{\alpha}_{0,n} - \alpha_0\|^2 \asymp Q(\overline{\alpha}_{0,n}) \asymp Q(\Pi_n\alpha_0)$ by assumption in the proposition.

Finally, the convergence rate under $\|\cdot\|_s$ is obtained by applying the previous result and the definition of τ_n. *Q.E.D.*

PROOF OF THEOREM 4.4: Since $\sup_{h\in\mathcal{H}} \operatorname{Pen}(h) < \infty$, the relevant parameter set is $\mathcal{A}^M \equiv \{\alpha \in \mathcal{A} : \operatorname{Pen}(h) \leq M\}$ with $M = \sup_{h\in\mathcal{H}} \operatorname{Pen}(h)$, which is nonempty and compact (in $\|\cdot\|_s$) under Assumptions 3.1(i)(ii) and 3.2(iii). Let $\mathcal{A}^{R,M} = \mathcal{A}^M \cap \{\alpha \in \mathcal{A} : \phi(\alpha) = \phi_0\}$. Since ϕ is continuous in $\|\cdot\|_s$, $\mathcal{A}^{R,M}$ is also compact (in $\|\cdot\|_s$). Note that $\alpha_0 \in \mathcal{A}^{R,M}$ iff the null $H_0 : \phi(\alpha_0) = \phi_0$ holds.

If $\mathcal{A}^{R,M}$ is empty, then there does not exist any $\alpha \in \mathcal{A}^M$ such that $\phi(\alpha) = \phi_0$, and hence it holds trivially that $\widehat{\mathrm{QLR}}_n(\phi_0) \geq nC$ for some $C > 0$ wpa1.

If $\mathcal{A}^{R,M}$ is non-empty, under Assumption 3.1(iii) we have: $\min_{\alpha\in\mathcal{A}^{R,M}} Q(\alpha)$ is achieved at some point within $\mathcal{A}^{R,M}$, say, $\overline{\alpha} \in \mathcal{A}^{R,M}$. This and Assumption 3.1(i)(iv) imply that $Q(\overline{\alpha}) = \min_{\alpha\in\mathcal{A}^{R,M}} Q(\alpha) > 0 = Q(\alpha_0)$ under the fixed alternatives $H_1 : \phi(\alpha_0) \neq \phi_0$.

By definitions of $\widehat{\alpha}_n$ and $\Pi_n\alpha_0$ and Assumption 3.3(i), we have

$$\widehat{Q}_n(\widehat{\alpha}_n) \leq \widehat{Q}_n(\Pi_n\alpha_0) \leq c_0 Q(\Pi_n\alpha_0) + o_{P_{Z^\infty}}\big(n^{-1}\big).$$

Since $M = \sup_{h\in\mathcal{H}} \operatorname{Pen}(h) < \infty$, we also have that $\widehat{\alpha}_n^R \in \mathcal{A}_{k(n)}^{R,M} \subseteq \mathcal{A}_{k(n)}^M$ wpa1, so by Assumption 3.3(ii), we have

$$\widehat{Q}_n\big(\widehat{\alpha}_n^R\big) \geq cQ\big(\widehat{\alpha}_n^R\big) - O_{P_{Z^\infty}}\big(\overline{\delta}_{m,n}^2\big) \geq c \times \min_{\alpha\in\mathcal{A}^{R,M}} Q(\alpha) - O_{P_{Z^\infty}}\big(\overline{\delta}_{m,n}^2\big).$$

Thus

$$\begin{aligned}
&\widehat{Q}_n(\widehat{\alpha}_n^R) - \widehat{Q}_n(\widehat{\alpha}_n) \\
&\quad \geq c \times \min_{\alpha \in \mathcal{A}^{R,M}} Q(\alpha) - c_0 Q(\Pi_n \alpha_0) - o_{P_{Z^\infty}}(n^{-1}) - O_{P_{Z^\infty}}(\overline{\delta}^2_{m,n}) \\
&\quad = cQ(\overline{\alpha}) + o_{P_{Z^\infty}}(1).
\end{aligned}$$

Thus under the fixed alternatives $H_1: \phi(\alpha_0) \neq \phi_0$,

$$\frac{\widehat{\mathrm{QLR}}_n(\phi_0)}{n} \geq cQ(\overline{\alpha}) > 0 \quad \text{wpa1} .$$ *Q.E.D.*

A consistent variance estimator for optimally weighted PSMD estimator. To stress the fact that we consider the optimally weighted PSMD procedure, we use v_n^0 and $\|v_n^0\|_0$ to denote the corresponding v_n^* and $\|v_n^*\|$ computed using the optimal weighting matrix $\Sigma = \Sigma_0$. That is,

$$\|v_n^0\|_0^2 = E\left[\left(\frac{dm(X,\alpha_0)}{d\alpha}[v_n^0]\right)' \Sigma_0(X)^{-1}\left(\frac{dm(X,\alpha_0)}{d\alpha}[v_n^0]\right)\right].$$

We call the corresponding sieve score, $S_{n,i}^0 \equiv (\frac{dm(X_i,\alpha_0)}{d\alpha}[v_n^0])'\Sigma_0(X_i)^{-1}\rho(Z_i,\alpha_0)$, the optimal sieve score. Note that $\|v_n^0\|_{\mathrm{sd}}^2 = \mathrm{Var}(S_{n,i}^0) = \|v_n^0\|_0^2$. By Theorem 4.1, $\|v_n^0\|_{\mathrm{sd}}^2 = \|v_n^0\|_0^2$ is the variance of the optimally weighted PSMD estimator $\phi(\widehat{\alpha}_n)$. We could compute a consistent estimator $\widehat{\|v_n^0\|_0^2}$ of the variance $\|v_n^0\|_0^2$ by looking at the "slope" of the optimally weighted criterion $\widehat{Q}_n^0$:

$$\widehat{\|v_n^0\|_0^2} \equiv \left(\frac{\widehat{Q}_n^0(\widetilde{\alpha}_n) - \widehat{Q}_n^0(\widehat{\alpha}_n)}{\varepsilon_n^2}\right)^{-1}, \tag{B.13}$$

where $\widetilde{\alpha}_n$ is an approximate minimizer of $\widehat{Q}_n^0(\alpha)$ over $\{\alpha \in \mathcal{A}_{k(n)}: \phi(\alpha) = \phi(\widehat{\alpha}_n) - \varepsilon_n\}$.

THEOREM B.2: *Let $\widehat{\alpha}_n$ be the optimally weighted PSMD estimator* (2.2) *with $\Sigma = \Sigma_0$, and conditions for Lemma* 3.2, *Assumptions* 3.5 *and* 3.6 *hold with $\|v_n^0\|_{\mathrm{sd}} = \|v_n^0\|_0$ and $|B_n - 1| = o_{P_{Z^\infty}}(1)$. Let $cn^{-1/2} \leq \frac{\varepsilon_n}{\|v_n^0\|_0} \leq C\delta_n$ for finite constants $c, C > 0$. Then: $\widetilde{\alpha}_n \in \mathcal{N}_{osn}$ wpa1-P_{Z^∞}, and*

$$\frac{\widehat{\|v_n^0\|_0^2}}{\|v_n^0\|_0^2} = 1 + o_{P_{Z^\infty}}(1).$$

When $\widehat{\alpha}_n$ is the optimally weighted PSMD estimator of α_0, Theorem B.2 suggests $\widehat{\|v_n^0\|_0^2}$ defined in (B.13) as an alternative consistent variance estimator for

$\phi(\widehat{\alpha}_n)$. Compared to Theorems 4.2 and B.1, this alternative variance estimator $\widehat{\|v_n^0\|_0^2}$ allows for a non-smooth residual function $\rho(Z,\alpha)$ (such as the one in NPQIV), but is only valid for an optimally weighted PSMD estimator.

PROOF OF THEOREM B.2: Recall that for the optimally weighted criterion case $u_n^* = v_n^0/\|v_n^0\|_0$, and hence $\|u_n^*\| = 1$, $B_n = 1 + o_{P_{Z^\infty}}(1)$. To simplify notation, in this proof we use $\langle\cdot,\cdot\rangle$, $\|\cdot\|$, and $\widehat{Q}_n(\cdot)$ for the ones corresponding to the optimal weighting matrix $\Sigma = \Sigma_0$.

We first show that $\widetilde{\alpha}_n \in \mathcal{N}_{osn}$ *wpa*1. Recall that $\widetilde{\alpha}_n$ is defined as an approximate optimally weighted PSMD estimator constrained to $\{\alpha \in \mathcal{A}_{k(n)} : \phi(\alpha) = \phi(\widehat{\alpha}_n) - \varepsilon_n\}$. In the following, since there is no risk of confusion, we use $_P$ instead of $_{P_{Z^\infty}}$.

Let $r = \phi(\widehat{\alpha}_n) - \phi(\alpha_0) - \varepsilon_n$. Since $\varepsilon_n \le C\|v_n^0\|_0\delta_n$ (by assumption), and $\widehat{\alpha}_n \in \mathcal{N}_{osn}$ wpa1, $\phi(\widehat{\alpha}_n) - \phi(\alpha_0) = O_P(\|v_n^0\|_0/\sqrt{n})$ (by Theorem 4.1), we have $|r| \le C\|v_n^0\|_0 O(\delta_n + n^{-1/2}) \le C\|v_n^0\|_0\delta_n$ for some $C > 0$. Also note that $\|\widehat{\alpha}_n - \alpha_0\| \le C\delta_n$ wpa1. Thus, by Lemma B.2(2), there exists a $t_n^* \in \mathcal{T}_n$ such that $\phi(\widehat{\alpha}_n(t_n^*)) = \phi(\widehat{\alpha}_n) - \varepsilon_n$ and $\widehat{\alpha}_n(t_n^*) = \widehat{\alpha}_n + t_n^* u_n^* \in \mathcal{A}_{k(n)}$ and $t_n^* = O(\delta_n)$. Henceforth, let $\overline{\alpha}_n \equiv \widehat{\alpha}_n(t_n^*)$. Observe that

$$\|\overline{\alpha}_n - \alpha_0\| \le \delta_n + t_n^* = O(\delta_n),$$

and

$$\|\overline{\alpha}_n - \alpha_0\|_s \le \|\widehat{\alpha}_n - \alpha_0\|_s + t_n^*\left\|u_n^*\right\|_s \le \delta_{s,n} + t_n^*\tau_n$$

which is of order $\delta_{s,n}$. Therefore, $\overline{\alpha}_n$ satisfies: (a) $\overline{\alpha}_n \in \mathcal{N}_{osn}$ wpa1, and $t_n^* \in \mathcal{T}_n$ with $t_n^* = O(\delta_n)$; and (b) $\overline{\alpha}_n \in \{\alpha \in \mathcal{A}_{k(n)} : \phi(\alpha) = \phi(\widehat{\alpha}_n) - \varepsilon_n\}$.

We now establish the consistency of $\widetilde{\alpha}_n$ using the properties of $\overline{\alpha}_n$. We observe that, for any $\epsilon > 0$,

$$\begin{aligned}
&\Pr\big(\|\widetilde{\alpha}_n - \alpha_0\|_s \ge \epsilon\big) \\
&\quad \le \Pr\Big(\inf_{\mathcal{B}_n : \|\alpha - \alpha_0\|_s \ge \epsilon} \widehat{Q}_n(\alpha) \le \widehat{Q}_n(\overline{\alpha}_n) + o\big(n^{-1}\big) + \lambda_n \operatorname{Pen}(\overline{h}_n)\Big)
\end{aligned}$$

where $\mathcal{B}_n \equiv \{\alpha \in \mathcal{A}_{k(n)}^{M_0} : \phi(\alpha) = \phi(\widehat{\alpha}_n) - \varepsilon_n\}$ and the inequality is valid because $\overline{\alpha}_n \in \mathcal{B}_n$ by (a) and (b). Under (a) and Lemma 3.2, $\lambda_n \operatorname{Pen}(\overline{h}_n) = O_P(\lambda_n) = o(n^{-1})$.

By (a), under assumption 3.6(i),

$$\begin{aligned}
\widehat{Q}_n(\overline{\alpha}_n) &= \widehat{Q}_n(\widehat{\alpha}_n) + t_n^*\big\{\mathbb{Z}_n + \big\langle u_n^*, \widehat{\alpha}_n - \alpha_0\big\rangle\big\} \\
&\qquad + 0.5\big(t_n^*\big)^2 + o_P\big(t_n^* n^{-1/2} + \big(t_n^*\big)^2 + o\big(n^{-1}\big)\big).
\end{aligned}$$

By Lemma B.1, $\mathbb{Z}_n + \langle u_n^*, \widehat{\alpha}_n - \alpha_0\rangle = o_P(n^{-1/2})$ and thus, given that $t_n^* = O(\delta_n)$, the previous display implies that

$$\widehat{Q}_n(\overline{\alpha}_n) \leq \widehat{Q}_n(\widehat{\alpha}_n) + o_P\big(n^{-1/2}\delta_n + \delta_n^2 + o(n^{-1})\big) \leq O_P\big(\delta_n^2\big).$$

Therefore,

$$\begin{aligned}&\Pr\big(\|\widetilde{\alpha}_n - \alpha_0\|_s \geq \epsilon\big)\\ &\quad \leq \Pr\Big(\inf_{\mathcal{B}_n:\|\alpha-\alpha_0\|_s\geq\epsilon} \widehat{Q}_n(\alpha) \leq \widehat{Q}_n(\widehat{\alpha}_n) + O\big(\lambda_n + \delta_n^2\big)\Big).\end{aligned}$$

Since $\widehat{Q}_n(\widehat{\alpha}_n) \leq \widehat{Q}_n(\Pi_n\alpha_0) + O(\lambda_n)$ by definition of $\widehat{\alpha}_n$ and from the fact that $\mathcal{B}_n \subseteq \mathcal{A}_{k(n)}^{M_0}$, it follows that

$$\begin{aligned}&\Pr\big(\|\widetilde{\alpha}_n - \alpha_0\|_s \geq \epsilon\big)\\ &\quad \leq \Pr\Big(\inf_{\mathcal{A}_n^{M_0}:\|\alpha-\alpha_0\|_s\geq\epsilon} \widehat{Q}_n(\alpha) \leq \widehat{Q}_n(\Pi_n\alpha_0) + O\big(\lambda_n + \delta_n^2\big)\Big).\end{aligned}$$

The rest of the consistency proof follows from identical steps to the standard one; see Chen and Pouzo (2009).

In order to show the rate, by similar arguments to the previous ones

$$\widehat{Q}_n(\widetilde{\alpha}_n) \leq \widehat{Q}_n(\Pi_n\alpha_0) + O\big(\lambda_n + \delta_n^2\big),$$

under our assumptions $\widehat{Q}_n(\widetilde{\alpha}_n) \geq c\|\widetilde{\alpha}_n - \alpha_0\|^2 - O_P(\delta_n^2)$ and $\widehat{Q}_n(\Pi_n\alpha_0) \leq c_0 Q(\Pi_n\alpha_0) + o_P(n^{-1})$, so the desired rate under $\|\cdot\|$ follows. The rate under $\|\cdot\|_s$ immediately follows using the definition of sieve measure of local ill-posedness τ_n. Thus $\widetilde{\alpha}_n \in \mathcal{N}_{osn}$ wpa1.

We now show that $\frac{\widehat{\|v_n^0\|_0^2}}{\|v_n^0\|_0^2} = 1 + o_{P_{Z^\infty}}(1)$. This part of the proof consists of several steps that are similar to those in the proof of Theorem 4.3, and hence we omit some details. We first provide an asymptotic expansion for $n(\widehat{Q}_n(\widetilde{\alpha}_n) - \widehat{Q}_n(\widehat{\alpha}_n))$ using Assumption 3.6(i) (with $B_n = 1 + o_{P_{Z^\infty}}(1)$), and then show that this is enough to establish the desired result.

In the following we let $t_n \equiv \varepsilon_n/\|v_n^0\|_0$. By the assumption on ε_n we have: $cn^{-1/2} \leq t_n \leq C\delta_n$. Therefore, $t_n \in \mathcal{T}_n$, $t_n = o_{P_{Z^\infty}}(1)$, and $o_{P_{Z^\infty}}(\frac{1}{t_n}n^{-1/2}) = o_{P_{Z^\infty}}(1)$.

STEP 1: First, we note that $\widehat{\alpha}_n \in \mathcal{N}_{osn}$ wpa1, that $-t_n \in \mathcal{T}_n$, and $\widehat{\alpha}_n(-t_n) \in \mathcal{A}_{k(n)}$. So we can apply Assumption 3.6(i) with $\alpha = \widehat{\alpha}_n$ and $-t_n$ as the direction, and obtain

$$\text{(B.14)}\qquad \frac{\big(\widehat{Q}_n\big(\widehat{\alpha}_n(-t_n)\big)-\widehat{Q}_n(\widehat{\alpha}_n)\big)}{t_n^2} = \frac{-2}{t_n}\big\{\mathbb{Z}_n+\big\langle u_n^*,\widehat{\alpha}_n-\alpha_0\big\rangle\big\}+1 + o_P\bigg(\max\bigg\{1,\frac{n^{-1/2}}{t_n},\frac{o\big(n^{-1}\big)}{t_n^2}\bigg\}\bigg) = 1+o_{P_{Z^\infty}}(1),$$

where the last equality follows from the fact that $\langle u_n^*,\widehat{\alpha}_n-\alpha_0\rangle+\mathbb{Z}_n = o_{P_{Z^\infty}}(n^{-1/2})$ (by Lemma B.1), and that $o_{P_{Z^\infty}}(\frac{1}{t_n}n^{-1/2}) = o_{P_{Z^\infty}}(1)$ (by our choice of t_n).

STEP 2: Since $\widetilde{\alpha}_n\in\mathcal{N}_{osn}$ wpa1, $t_n\in\mathcal{T}_n$, and $\widetilde{\alpha}_n(t_n)\in\mathcal{A}_{k(n)}$, we can apply Assumption 3.6(i) with $\alpha=\widetilde{\alpha}_n$ and t_n as the direction, and obtain

$$\text{(B.15)}\qquad \frac{\big(\widehat{Q}_n\big(\widetilde{\alpha}_n(t_n)\big)-\widehat{Q}_n(\widetilde{\alpha}_n)\big)}{t_n^2} = \frac{2}{t_n}\big\{\mathbb{Z}_n+\big\langle u_n^*,\widetilde{\alpha}_n-\alpha_0\big\rangle\big\}+1 + o_P\bigg(\max\bigg\{1,\frac{n^{-1/2}}{t_n},\frac{o\big(n^{-1}\big)}{t_n^2}\bigg\}\bigg) = -1+o_{P_{Z^\infty}}(1),$$

where the last line follows from the definition of the restricted estimator $\widetilde{\alpha}_n$. This is because $\phi(\widetilde{\alpha}_n)=\phi(\widehat{\alpha}_n)-\varepsilon_n$, by Assumptions 3.5(i)(ii),

$$\left|-\varepsilon_n-\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n-\widetilde{\alpha}_n]\right| = o_{P_{Z^\infty}}\big(\big\|v_n^0\big\|_0/\sqrt{n}\big).$$

Hence $\langle v_n^0,\widetilde{\alpha}_n-\alpha_0\rangle = \langle v_n^0,\widehat{\alpha}_n-\alpha_0\rangle-\varepsilon_n+o_{P_{Z^\infty}}(\|v_n^0\|_0/\sqrt{n})$. This implies that $\mathbb{Z}_n+\langle u_n^*,\widetilde{\alpha}_n-\alpha_0\rangle = -\frac{\varepsilon_n}{\|v_n^0\|_0}+o_{P_{Z^\infty}}(n^{-1/2}) = -t_n+o_{P_{Z^\infty}}(n^{-1/2})$.

STEP 3: It is easy to see that, from equation (B.15) and by the definition of $\widehat{\alpha}_n$,

$$\frac{\big(\widehat{Q}_n(\widetilde{\alpha}_n)-\widehat{Q}_n(\widehat{\alpha}_n)\big)}{t_n^2} \geq \frac{\big(\widehat{Q}_n(\widetilde{\alpha}_n)\big)-\widehat{Q}_n\big(\widetilde{\alpha}_n(t_n)\big)}{t_n^2}-o_{P_{Z^\infty}}(1) = 1+o_{P_{Z^\infty}}(1).$$

Also, from equation (B.14), Assumption 3.6(i), and by the definition of $\widetilde{\alpha}_n$,

$$\frac{\big(\widehat{Q}_n(\widetilde{\alpha}_n)-\widehat{Q}_n(\widehat{\alpha}_n)\big)}{t_n^2} \leq \frac{\big(\widehat{Q}_n\big(\widehat{\alpha}_n\big(t_n^*\big)\big)-\widehat{Q}_n(\widehat{\alpha}_n)\big)}{t_n^2}+o_{P_{Z^\infty}}(1)$$

$$
\begin{aligned}
&= \frac{2t_n^*\{\mathbb{Z}_n + \langle u_n^*, \widehat{\alpha}_n - \alpha_0\rangle\} + (t_n^*)^2}{t_n^2} \\
&\quad + t_n^{-2} o_P\big(\max\{(t_n^*)^2, t_n^* n^{-1/2}, o(n^{-1})\}\big) + o_P(1) \\
&= \frac{-2}{t_n}\{\mathbb{Z}_n + \langle u_n^*, \widehat{\alpha}_n - \alpha_0\rangle\} + 1 + o_{P_{Z^\infty}}(1) \\
&= 1 + o_{P_{Z^\infty}}(1),
\end{aligned}
$$

provided that there is a $t_n^* \in \mathcal{T}_n$ such that (3a) $\phi(\widehat{\alpha}_n(t_n^*)) = \phi(\widehat{\alpha}_n) - \varepsilon_n$ and (3b) $t_n^*/t_n = -1 + o_{P_{Z^\infty}}(1)$. In Step 5, we verify that such a t_n^* exists.

By putting these inequalities together, it follows that

$$
\text{(B.16)} \qquad \|v_n^0\|_0^2 \frac{\widehat{Q}_n(\widetilde{\alpha}_n) - \widehat{Q}_n(\widehat{\alpha}_n)}{\varepsilon_n^2} = \frac{\big(\widehat{Q}_n(\widetilde{\alpha}_n) - \widehat{Q}_n(\widehat{\alpha}_n)\big)}{t_n^2} = 1 + o_{P_{Z^\infty}}(1).
$$

STEP 4: By equation (B.16) we have

$$
\frac{\|v_n^0\|_0^2}{\widehat{\|v_n^0\|_0^2}} = 1 + o_{P_{Z^\infty}}(1), \quad \text{with} \quad \widehat{\|v_n^0\|_0^2} \equiv \left(\frac{\widehat{Q}_n(\widetilde{\alpha}_n) - \widehat{Q}_n(\widehat{\alpha}_n)}{\varepsilon_n^2}\right)^{-1},
$$

which implies that $0.5 \leq \frac{\|v_n^0\|_0^2}{\widehat{\|v_n^0\|_0^2}} \leq 1.5$ with probability P_{Z^∞} approaching 1. By the continuous mapping theorem, we obtain

$$
\frac{\widehat{\|v_n^0\|_0^2}}{\|v_n^0\|_0^2} = 1 + o_{P_{Z^\infty}}(1).
$$

STEP 5: We now show that there is a $t_n^* \in \mathcal{T}_n$ such that (3a) and (3b) in Step 3 hold. Denote $r \equiv \phi(\widehat{\alpha}_n) - \phi(\alpha_0) - \varepsilon_n$. Since $\varepsilon_n \leq C\|v_n^0\|_0\delta_n$, and $\widehat{\alpha}_n \in \mathcal{N}_{osn}$ wpa1, $\phi(\widehat{\alpha}_n) - \phi(\alpha_0) = O_P(\|v_n^0\|_0/\sqrt{n})$ (by Theorem 4.1), we have $|r| \leq \|v_n^0\|_0\delta_n(M_n + C) \leq 2M_n\|v_n^0\|_0\delta_n$ (since $C < M_n$ eventually). Thus, by Lemma B.2, there exists a $t_n^* \in \mathcal{T}_n$ such that $\phi(\widehat{\alpha}_n(t_n^*)) = \phi(\widehat{\alpha}_n) - \varepsilon_n$ and $\widehat{\alpha}_n(t_n^*) = \widehat{\alpha}_n + t_n^* u_n^* \in \mathcal{A}_{k(n)}$, and hence (3a) holds. Moreover, by Assumption 3.5(i)(ii), such a choice of t_n^* also satisfies

$$
\Bigg|\underbrace{\phi\big(\widehat{\alpha}_n(t_n^*)\big) - \phi(\widehat{\alpha}_n)}_{=-\varepsilon_n} - \frac{d\phi(\alpha_0)}{d\alpha}[t_n^* u_n^*]\Bigg| = o_{P_{Z^\infty}}\big(\|v_n^0\|_0 n^{-1/2}\big).
$$

Since $u_n^* = v_n^0/\|v_n^0\|_0$ for optimally weighted criterion case, we have $\frac{d\phi(\alpha_0)}{d\alpha}[u_n^*] = \|v_n^0\|_0$. Thus

$$
\big|-\varepsilon_n - t_n^*\|v_n^0\|_0\big| = o_{P_{Z^\infty}}\big(\|v_n^0\|_0 n^{-1/2}\big).
$$

Since $t_n \equiv \varepsilon_n / \|v_n^0\|_0$, we obtain $|-t_n - t_n^*| = o_{P_{Z^\infty}}(n^{-1/2})$, and hence

$$\left|\left(t_n^*/t_n\right) + 1\right| = o_{P_{Z^\infty}}\left(n^{-1/2}/t_n\right) = o_{P_{Z^\infty}}(1)$$

due to the fact that $cn^{-1/2} \leq t_n \leq C\delta_n$. Thus (3b) holds. *Q.E.D.*

B.3. *Proofs for Section 5 on Bootstrap Inference*

Throughout the appendices, we sometimes use the simplified term "wpa1" in the bootstrap world while its precise meaning is given in Section 5.

Recall that $\mathbb{Z}_n^\omega \equiv \frac{1}{n}\sum_{i=1}^n \omega_i g(X_i, u_n^*)\rho(Z_i, \alpha_0)$ with $g(X_i, u_n^*) \equiv (\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*])'\Sigma(X_i)^{-1}$.

LEMMA B.3: *Let* $\widehat{\alpha}_n^B$ *be the bootstrap PSMD estimator and conditions for Lemma* 3.2 *and Lemma* A.1 *hold*. *Let Assumption* Boot.3(i) *hold*. *Then*: (1) *for all* $\delta > 0$, *there exists a* $N(\delta)$ *such that*, *for all* $n \geq N(\delta)$,

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sqrt{n}\left|\langle u_n^*, \widehat{\alpha}_n^B - \alpha_0\rangle + \mathbb{Z}_n^\omega\right| \geq \delta | Z^n\right) < \delta\right) \geq 1 - \delta.$$

(2) *If*, *in addition*, *assumptions of Lemma* B.1 *hold*, *then*

$$\sqrt{n}\langle u_n^*, \widehat{\alpha}_n^B - \widehat{\alpha}_n\rangle = -\sqrt{n}\mathbb{Z}_n^{\omega-1} + o_{P_{V^\infty|Z^\infty}}(1) \quad wpa1\ (P_{Z^\infty}).$$

PROOF: The proof is very similar to that of Lemma B.1, so we only present the main steps.

For *Result (1)*. Under Assumption Boot.3(i) and using the fact that $\widehat{\alpha}_n^B$ is an approximate minimizer of $\widehat{Q}_n^B(\alpha) + \lambda_n \operatorname{Pen}(h)$ on $\mathcal{A}_{k(n)}$ and $\widehat{\alpha}_n^B \in \mathcal{N}_{osn}$ wpa1, it follows (see the proof of Lemma B.1 for details), for sufficiently large n,

$$\begin{aligned} &P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(2\epsilon_n\left\{\mathbb{Z}_n^\omega + \langle u_n^*, \widehat{\alpha}_n^B - \alpha_0\rangle\right\} + \epsilon_n^2 B_n^\omega + E_n\left(\widehat{\alpha}_n^B, \epsilon_n\right)\right.\right. \\ &\quad \left.\left. \geq -\delta r_n^{-1} | Z^n\right) \geq 1 - \delta\right) > 1 - \delta, \end{aligned}$$

where r_n and E_n are defined as in the proof of Lemma B.1, and $\epsilon_n = \pm\{s_n^{-1/2} + o(n^{-1/2})\}$. Dividing by $2\epsilon_n$ and multiplying by $\sqrt{n}$, it follows that

$$\begin{aligned} &P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(A_{n,\delta}^\omega \geq \sqrt{n}\left\{\mathbb{Z}_n^\omega + \langle u_n^*, \widehat{\alpha}_n^B - \alpha_0\rangle\right\} \geq B_{n,\delta}^\omega | Z^n\right) \geq 1 - \delta\right) \\ &\quad > 1 - \delta \end{aligned}$$

eventually, where

$$\begin{aligned} A_{n,\delta}^\omega &\equiv -0.5\sqrt{n}\epsilon_n B_n^\omega - \delta\sqrt{n}\epsilon_n^{-1} r_n^{-1} + 0.5\delta, \\ B_{n,\delta}^\omega &\equiv -0.5\sqrt{n}\epsilon_n B_n^\omega - \delta\sqrt{n}\epsilon_n^{-1} r_n^{-1} - 0.5\delta. \end{aligned}$$

Since $\sqrt{n}\epsilon_n = o(1)$ and $B_n^{\omega} = O_{P_{V^\infty|Z^\infty}}(1)$ wpa1 (P_{Z^∞}) and $|\sqrt{n}\epsilon_n^{-1} r_n^{-1}| \asymp 1$, it follows, for sufficiently large n,

$$P_{Z^\infty}\big(P_{V^\infty|Z^\infty}\big(2\delta \geq \sqrt{n}\big\{\mathbb{Z}_n^{\omega} + \langle u_n^*, \widehat{\alpha}_n^B - \alpha_0\rangle\big\} \geq -2\delta|Z^n\big) \geq 1-\delta\big) > 1-\delta.$$

Or equivalently, for sufficiently large n,

$$P_{Z^\infty}\big(P_{V^\infty|Z^\infty}\big(\big|\sqrt{n}\big\{\mathbb{Z}_n^{\omega} + \langle u_n^*, \widehat{\alpha}_n^B - \alpha_0\rangle\big\}\big| \geq 2\delta|Z^n\big) < \delta\big) \geq 1-\delta.$$

Result (2) directly follows from Result (1) and Lemma B.1. *Q.E.D.*

PROOF OF THEOREM 5.1: We note that Assumption Boot.4 implies that $|n^{-1}\sum_{i=1}^n \widehat{T}_i[v_n]'\widehat{M}_i^B\widehat{T}_i[v_n] - \sigma_\omega^2 n^{-1}\sum_{i=1}^n \widehat{T}_i[v_n]'\widehat{M}_i\widehat{T}_i[v_n]| = o_{P_{V^\infty|Z^\infty}}(1)$, uniformly over $v_n \in \overline{\mathbf{V}}_{k(n)}^1$ with $\widehat{M}_i = \widehat{M}(Z_i, \widehat{\alpha}_n)$ and $\widehat{T}_i[v_n] \equiv \frac{d\widehat{m}(X_i,\widehat{\alpha}_n)}{d\alpha}[v_n]$. The rest of the proof follows directly from that of Theorem 4.2(1) for the sieve variance defined in (4.7) case. *Q.E.D.*

PROOF OF THEOREM 5.2: By Lemma B.3 and steps analogous to those used to show Theorem 4.1, it follows that

$$\sqrt{n}\frac{\phi(\widehat{\alpha}_n^B) - \phi(\widehat{\alpha}_n)}{\sigma_\omega \|v_n^*\|_{\mathrm{sd}}} = -\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega} + o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{Z^\infty}). \tag{B.17}$$

For *Result (1)*, we note that the result for $\widehat{W}_{2,n}^B$ follows directly from Theorem 5.1 and the proof of the Result (1) for $\widehat{W}_{1,n}^B \equiv \sqrt{n}\frac{\phi(\widehat{\alpha}_n^B)-\phi(\widehat{\alpha}_n)}{\sigma_\omega\|\widehat{v}_n^*\|_{n,\mathrm{sd}}}$.

In fact, for both $j = 1, 2$, Theorem 4.2(1), equation (B.17), and Theorem 5.1 imply that

$$\widehat{W}_{j,n}^B = -\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega} + o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{Z^\infty}); \tag{B.18}$$

equation (B.18) and Assumptions 3.6(ii) and Boot.3(ii) imply that

$$\big|\mathcal{L}_{V^\infty|Z^\infty}\big(\widehat{W}_{j,n}^B|Z^n\big) - \mathcal{L}(\widehat{W}_n)\big| = o_{P_{Z^\infty}}(1).$$

Result (1) now follows from the following two equations:

$$\sup_{t\in\mathbb{R}}\big|P_{V^\infty|Z^\infty}\big(\widehat{W}_{j,n}^B \leq t|Z^n\big) - \Phi(t)\big| = o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{Z^\infty}), \tag{B.19}$$

and

$$\sup_{t\in\mathbb{R}}\big|P_{Z^\infty}(\widehat{W}_n \leq t) - \Phi(t)\big| = o_{P_{Z^\infty}}(1), \tag{B.20}$$

where $\Phi()$ is the c.d.f. of a standard normal. Equation (B.20) follows directly from Theorem 4.2(2) and Polya's theorem (see, e.g., Bickel and Millar (1992)). Equation (B.19) follows by the same arguments in Lemma 10.11 in Kosorok (2008) (which are in turn analogous to those used in the proof of Polya's theorem).

Result (2) follows from equation (B.17) and the fact that $\|v_n^*\|_{sd} \to \|v^*\|_{sd} \in (0,\infty)$ for regular functionals. *Q.E.D.*

PROOF OF THEOREM 5.3: For *Result (1)*, denote

$$\mathcal{F}_n \equiv n\frac{\inf_{\mathcal{A}_{k(n)}(\widehat{\phi}_n)} \widehat{Q}_n^B(\alpha) - \widehat{Q}_n^B(\widehat{\alpha}_n^B)}{\sigma_\omega^2} = \frac{\widehat{\mathrm{QLR}}_n^B(\widehat{\phi}_n)}{\sigma_\omega^2}$$
$$= n\frac{\widehat{Q}_n^B(\widehat{\alpha}_n^{R,B}) - \widehat{Q}_n^B(\widehat{\alpha}_n^B)}{\sigma_\omega^2} + o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{Z^\infty}),$$

where $\mathcal{A}_{k(n)}(\widehat{\phi}_n) \equiv \{\alpha \in \mathcal{A}_{k(n)} : \phi(\alpha) = \phi(\widehat{\alpha}_n)\}$. Since $o_{P_{V^\infty|Z^\infty}}(1)$ wpa1 (P_{Z^∞}) will not affect the asymptotic results, we omit it from the rest of the proof to ease the notational burden. We want to show that for all $\delta > 0$, there exists a $N(\delta)$ such that

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\left|\mathcal{F}_n - \left(\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega\|u_n^*\|}\right)^2\right| \geq \delta \Big| Z^n\right) < \delta\right) \geq 1-\delta$$

for all $n \geq N(\delta)$. We divide the proof in several steps.

STEP 1: By assumption $|B_n^\omega - \|u_n^*\|^2| = o_{P_{V^\infty|Z^\infty}}(1)$ wpa1 (P_{Z^∞}) and $\|u_n^*\| \in (c, C)$, we have: $|\frac{\|u_n^*\|^2}{B_n^\omega} - 1| = o_{P_{V^\infty|Z^\infty}}(1)$ wpa1 (P_{Z^∞}). Therefore, it suffices to show that

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\left|\mathcal{F}_n - \left(\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega\sqrt{B_n^\omega}}\right)^2\right| \geq \delta \Big| Z^n\right) < \delta\right) \geq 1-\delta \tag{B.21}$$

eventually.

STEP 2: By Assumption Boot.3(i), for all $\delta > 0$, there is a $M > 0$ such that

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sqrt{n}\left|\mathbb{Z}_n^{\omega-1}/B_n^\omega\right| \geq M|Z^n\right) < \delta\right) \geq 1-\delta$$

eventually. Thus $t_n = -\mathbb{Z}_n^{\omega-1}/B_n^\omega \in \mathcal{T}_n$ wpa1. By the definition of $\widehat{\alpha}_n^B$, and the fact that $\widehat{\alpha}_n^{R,B} \in \mathcal{N}_{osn}$ wpa1 (by Lemma A.1(3)),

$$\mathcal{F}_n \geq n\frac{\widehat{Q}_n^B(\widehat{\alpha}_n^{R,B}) - \widehat{Q}_n^B(\widehat{\alpha}_n^{R,B}(t_n))}{\sigma_\omega^2} - o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{Z^\infty}).$$

By specializing Assumption Boot.3(i) to $\alpha = \widehat{\alpha}_n^{R,B}$ and $t_n = -\mathbb{Z}_n^{\omega-1}/B_n^{\omega}$, it follows that

$$0.5\left(\widehat{Q}_n^B\left(\widehat{\alpha}_n^{R,B}\left(-\frac{\mathbb{Z}_n^{\omega-1}}{B_n^{\omega}}\right)\right) - \widehat{Q}_n^B\left(\widehat{\alpha}_n^{R,B}\right)\right) = -\frac{\mathbb{Z}_n^{\omega-1}}{B_n^{\omega}}\left\{\mathbb{Z}_n^{\omega} + \langle u_n^*, \widehat{\alpha}_n^{R,B} - \alpha_0\rangle\right\} + \frac{\left(\mathbb{Z}_n^{\omega-1}\right)^2}{2B_n^{\omega}} + o_{P_{V^\infty|Z^\infty}}\left(r_n^{-1}\right) \quad \text{wpa1 } (P_{Z^\infty}). \tag{B.22}$$

By Assumption 3.5(i)(ii), and the fact that $\widehat{\alpha}_n^{R,B} \in \mathcal{N}_{osn}$ wpa1,

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\frac{\sqrt{n}}{\|v_n^*\|}\left|\underbrace{\phi\left(\widehat{\alpha}_n^{R,B}\right) - \phi\left(\widehat{\alpha}_n\right)}_{=0} - \frac{d\phi(\alpha_0)}{d\alpha}\left[\widehat{\alpha}_n^{R,B} - \widehat{\alpha}_n\right]\right| \geq \delta \,\middle|\, Z^n\right) < \delta\right) \geq 1 - \delta$$

eventually. Also by definition $\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n^{R,B} - \widehat{\alpha}_n] = \langle v_n^*, \widehat{\alpha}_n^{R,B} - \widehat{\alpha}_n\rangle$. This and Assumption 3.5(i) imply that

$$\sqrt{n}\langle u_n^*, \widehat{\alpha}_n^{R,B} - \widehat{\alpha}_n\rangle = o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{Z^\infty}). \tag{B.23}$$

Equation (B.23) and $\sqrt{n}\langle u_n^*, \widehat{\alpha}_n - \alpha_0\rangle = -\sqrt{n}\mathbb{Z}_n + o_{P_{Z^\infty}}(1)$ (Lemma B.1) imply that

$$\sqrt{n}\langle u_n^*, \widehat{\alpha}_n^{R,B} - \alpha_0\rangle = -\sqrt{n}\mathbb{Z}_n + o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{Z^\infty}).$$

Thus we can infer from equation (B.22) that

$$0.5\left(\widehat{Q}_n^B\left(\widehat{\alpha}_n^{R,B}\left(-\frac{\mathbb{Z}_n^{\omega-1}}{B_n^{\omega}}\right)\right) - \widehat{Q}_n^B\left(\widehat{\alpha}_n^{R,B}\right)\right) = -\frac{\left(\mathbb{Z}_n^{\omega-1}\right)^2}{2B_n^{\omega}} + o_{P_{V^\infty|Z^\infty}}\left(r_n^{-1}\right) \quad \text{wpa1 } (P_{Z^\infty}). \tag{B.24}$$

Since $nr_n^{-1} = O(1)$, multiplying both sides by $-2n\sigma_\omega^{-2}$, we obtain

$$\mathcal{F}_n \geq \left(\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega\sqrt{B_n^{\omega}}}\right)^2 - o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{Z^\infty}).$$

STEP 3: In order to show

$$\mathcal{F}_n \leq \left(\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega\sqrt{B_n^{\omega}}}\right)^2 + o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{Z^\infty}), \tag{B.25}$$

we can repeat the same calculations as in Step 2, provided there exists a $t_n^* \in \mathcal{T}_n$ wpa1 such that (a) $\phi(\widehat{\alpha}_n^B(t_n^*)) = \phi(\widehat{\alpha}_n)$ with $\widehat{\alpha}_n^B(t_n^*) \in \mathcal{A}_{k(n)}$, and (b) $t_n^* = Z_n^{\omega-1}/\|u_n^*\|^2 + o_{P_{V^\infty|Z^\infty}}(n^{-1/2}) = O_{P_{V^\infty|Z^\infty}}(n^{-1/2})$ wpa1 (P_{Z^∞}). This is because by (a) and the definition of $\widehat{\alpha}_n^{R,B}$,

$$n\frac{\widehat{Q}_n^B(\widehat{\alpha}_n^{R,B}) - \widehat{Q}_n^B(\widehat{\alpha}_n^B)}{\sigma_\omega^2} \le n\frac{\widehat{Q}_n^B(\widehat{\alpha}_n^B(t_n^*)) - \widehat{Q}_n^B(\widehat{\alpha}_n^B)}{\sigma_\omega^2} + o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{Z^\infty}).$$

By specializing Assumption Boot.3(i) to $\alpha = \widehat{\alpha}_n^B \in \mathcal{N}_{osn}$ wpa1 (by Lemma A.1(2)), and t_n^* as the direction, it follows that

$$\begin{aligned}
&0.5\big(\widehat{Q}_n^B(\widehat{\alpha}_n^B(t_n^*)) - \widehat{Q}_n^B(\widehat{\alpha}_n^B)\big) \\
&= t_n^*\big\{\mathbb{Z}_n^\omega + \langle u_n^*, \widehat{\alpha}_n^B - \alpha_0\rangle\big\} + \frac{B_n^\omega}{2}(t_n^*)^2 + o_{P_{V^\infty|Z^\infty}}(r_n^{-1}) \quad \text{wpa1 } (P_{Z^\infty}) \\
&= \frac{B_n^\omega}{2}\left(\frac{Z_n^{\omega-1}}{\|u_n^*\|^2} + o_{P_{V^\infty|Z^\infty}}(n^{-1/2})\right)^2 + o_{P_{V^\infty|Z^\infty}}(r_n^{-1}) \quad \text{wpa1 } (P_{Z^\infty}) \\
&= \frac{1}{2}\left(\frac{\mathbb{Z}_n^{\omega-1}}{\sqrt{B_n^\omega}}\right)^2 + o_{P_{V^\infty|Z^\infty}}(r_n^{-1}) \quad \text{wpa1 } (P_{Z^\infty}),
\end{aligned}$$

where the second equality is due to Lemma B.3(1) and (b), the third equality is due to the assumption $|B_n^\omega - \|u_n^*\|^2| = o_{P_{V^\infty|Z^\infty}}(1)$ wpa1 (P_{Z^∞}) and $\|u_n^*\| \in (c, C)$. Thus equation (B.25) holds.

STEP 4: We now show that there exists a t_n^* such that (a) and (b) hold in Step 3.

Let $r \equiv \phi(\widehat{\alpha}_n) - \phi(\alpha_0)$. Since $\widehat{\alpha}_n^B \in \mathcal{N}_{osn}$ wpa1, and $\phi(\widehat{\alpha}_n) - \phi(\alpha_0) = O_{P_{Z^\infty}}(\|v_n^*\|/\sqrt{n})$, by Lemma B.2, there is a $t_n^* \in \mathcal{T}_n$ wpa1 satisfying (a) with $\widehat{\alpha}_n^B(t_n^*) = \widehat{\alpha}_n^B + t_n^* u_n^* \in \mathcal{A}_{k(n)}$ and $\phi(\widehat{\alpha}_n^B(t_n^*)) - \phi(\alpha_0) = r$. Moreover, by Assumption 3.5(i)(ii), such a choice of t_n^* also satisfies

$$\left|\underbrace{\phi(\widehat{\alpha}_n^B(t_n^*)) - \phi(\widehat{\alpha}_n)}_{=0} - \frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n^B - \widehat{\alpha}_n + t_n^* u_n^*]\right| = o_{P_{V^\infty|Z^\infty}}(\|v_n^*\|/\sqrt{n}) \quad \text{wpa1 } (P_{Z^\infty}).$$

Thus, for sufficiently large n,

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\frac{\sqrt{n}}{\|v_n^*\|}\left|\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n^B-\widehat{\alpha}_n]+t_n^*\frac{\|v_n^*\|^2}{\|v_n^*\|_{\mathrm{sd}}}\right|\geq\delta\Big|Z^n\right)<\delta\right)$$
$$\geq 1-\delta.$$

By Assumption 3.5(i) and Lemma B.3(2), it follows that the LHS of the above equation is majorized by

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\frac{\sqrt{n}}{\|v_n^*\|}\left|\langle v_n^*,\widehat{\alpha}_n^B-\widehat{\alpha}_n\rangle+t_n^*\frac{\|v_n^*\|^2}{\|v_n^*\|_{\mathrm{sd}}}\right|\geq 2\delta\Big|Z^n\right)<\delta\right)+\delta$$
$$=P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\frac{\sqrt{n}}{\|v_n^*\|}\left|-\mathbb{Z}_n^{\omega-1}\|v_n^*\|_{\mathrm{sd}}+t_n^*\frac{\|v_n^*\|^2}{\|v_n^*\|_{\mathrm{sd}}}\right|\geq 2\delta\Big|Z^n\right)<\delta\right)$$
$$+\delta.$$

Therefore,

$$\sqrt{n}t_n^*=\sqrt{n}Z_n^{\omega-1}/\|u_n^*\|^2+o_{P_{V^\infty|Z^\infty}}(1)\quad\text{wpa1 }(P_{Z^\infty}).$$

Since $\sqrt{n}Z_n^{\omega-1}=O_{P_{V^\infty|Z^\infty}}(1)$ with probability P_{Z^∞} approaching 1 (Assumption Boot.3(ii)) and $\|u_n^*\|^2=O(1)$, we have $t_n^*=O_{P_{V^\infty|Z^\infty}}(n^{-1/2})$ with probability P_{Z^∞} approaching 1. Thus (b) holds.

Before we prove *Result (2)*, we wish to establish the following *equation* (B.26):

$$\left|\mathcal{L}_{V^\infty|Z^\infty}\left(\frac{\widehat{\mathrm{QLR}}_n^B(\widehat{\phi}_n)}{\sigma_\omega^2}\Big|Z^n\right)-\mathcal{L}\big(\widehat{\mathrm{QLR}}_n(\phi_0)|H_0\big)\right|=o_{P_{Z^\infty}}(1),\tag{B.26}$$

where $\mathcal{L}(\widehat{\mathrm{QLR}}_n(\phi_0)|H_0)$ denotes the law of $\widehat{\mathrm{QLR}}_n(\phi_0)$ under the null $H_0:\phi(\alpha)=\phi_0$, which will be simply denoted as $\mathcal{L}(\widehat{\mathrm{QLR}}_n(\phi_0))$ in the rest of the proof. By Result (1), it suffices to show that for any $\delta>0$, there exists a $N(\delta)$ such that

$$P_{Z^\infty}\left(\sup_{f\in BL_1}\left|E\left[f\left(\left[\frac{\sqrt{n}\mathbb{Z}_n^{\omega-1}}{\sigma_\omega\|u_n^*\|}\right]^2\right)\Big|Z^n\right]-E\big[f\big(\widehat{\mathrm{QLR}}_n(\phi_0)\big)\big]\right|\leq\delta\right)$$
$$\geq 1-\delta$$

for all $n \geq N(\delta)$. Let $\mathbb{Z}$ denote a standard normal random variable (i.e., $\mathbb{Z} \sim N(0,1)$). If the following equation (B.27) holds, which will be shown at the end of the proof of equation (B.26):

$$T_n \equiv \sup_{f \in BL_1} \left| E\left[f\left(\left[\frac{\mathbb{Z}}{\|u_n^*\|} \right]^2 \right) \right] - E\big[f\big(\widehat{\mathrm{QLR}}_n(\phi_0) \big) \big] \right| = o(1), \tag{B.27}$$

then, it suffices to show that

$$P_{Z^\infty}\left(\sup_{f \in BL_1} \left| E\left[f\left(\left[\frac{\sqrt{n}\mathbb{Z}_n^{\omega-1}}{\sigma_\omega \|u_n^*\|} \right]^2 \right) \Big| Z^n \right] - E\left[f\left(\left[\frac{\mathbb{Z}}{\|u_n^*\|} \right]^2 \right) \right] \right| \leq \delta \right) \geq 1 - \delta \tag{B.28}$$

for all $n \geq N(\delta)$. Suppose we could show that

$$\sup_{f \in BL_1} \left| E\left[f\left(\sqrt{n} \frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega \|u_n^*\|} \right) \Big| Z^n \right] - E\big[f\big(\mathbb{Z}\|u_n^*\|^{-1} \big) \big] \right| \to 0, \quad \text{wpa1 } (P_{Z^\infty}), \tag{B.29}$$

or equivalently,

$$P_{Z^\infty}\left(\left| \mathcal{L}_{V^\infty|Z^\infty}\left(\sqrt{n} \frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega \|u_n^*\|} \Big| Z^n \right) - \mathcal{L}\big(\mathbb{Z}\|u_n^*\|^{-1} \big) \right| \leq \delta \right) \geq 1 - \delta, \quad \text{eventually.}$$

Then, by the continuous mapping theorem (see Kosorok (2008), Theorem 10.8 and the discussion in Section 10.1.4), we have

$$P_{Z^\infty}\left(\left| \mathcal{L}_{V^\infty|Z^\infty}\left(\left(\sqrt{n} \frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega \|u_n^*\|} \right)^2 \Big| Z^n \right) - \mathcal{L}\big(\big(\mathbb{Z}\|u_n^*\|^{-1} \big)^2 \big) \right| \leq \delta \right) \geq 1 - \delta, \quad \text{eventually,}$$

and hence equation (B.28) follows.

It remains to show equation (B.29). By Assumption Boot.3(ii), and the fact that if a sequence converges in probability, for all subsequence, there exists a subsubsequence that converges almost surely, it follows for all subsequence $(n_k)_k$, there exists a subsubsequence $(n_{k(j)})_j$ such that

$$\left| \mathcal{L}_{V^\infty|Z^\infty}\left(\sqrt{n_{k(j)}} \frac{\mathbb{Z}_{n_{k(j)}}^{\omega-1}}{\sigma_\omega} \Big| Z^{n_{k(j)}} \right) - \mathcal{L}(\mathbb{Z}) \right| \to 0, \quad \text{a.s.-}P_{Z^\infty}.$$

Since $\|u_{n_{k(j)}}^*\| \in (c, C)$, then there exists a further subsequence (which we still denote as $n_{k(j)}$), such that $\lim_{j\to\infty} \|u_{n_{k(j)}}^*\| = d_\infty \in [c, C]$. Also, since $\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega}$ is a real-valued sequence, by Helly's theorem, convergence in distribution also holds for $(n_{k(j)})_j$. Therefore, by the Slutsky theorem,

$$\mathcal{L}_{V^\infty|Z^\infty}\left(\sqrt{n_{k(j)}}\frac{\mathbb{Z}^{\omega-1}_{n_{k(j)}}}{\sigma_\omega\|u^*_{n_{k(j)}}\|}\Big|Z^{n_{k(j)}}\right)-\mathcal{L}\left(\mathbb{Z}d_\infty^{-1}\right)\to 0,\quad \text{a.s.-}P_{Z^\infty}.$$

Since $\lim_{j\to\infty}\|u^*_{n_{k(j)}}\|=d_\infty\in[c,C]$ and $\mathbb{Z}$ is bounded in probability, this readily implies

$$\mathcal{L}_{V^\infty|Z^\infty}\left(\sqrt{n_{k(j)}}\frac{\mathbb{Z}^{\omega-1}_{n_{k(j)}}}{\sigma_\omega\|u^*_{n_{k(j)}}\|}\Big|Z^{n_{k(j)}}\right)-\mathcal{L}\left(\mathbb{Z}\|u^*_{n_{k(j)}}\|^{-1}\right)\to 0,\quad \text{a.s.-}P_{Z^\infty}.$$

Therefore, it follows that

$$\sup_{f\in BL_1}\left|E\left[f\left(\sqrt{n_{k(j)}}\frac{\mathbb{Z}^{\omega-1}_{n_{k(j)}}}{\sigma_\omega\|u^*_{n_{k(j)}}\|}\right)\Big|Z^{n_{k(j)}}\right]-E\left[f\left(\mathbb{Z}\|u^*_{n_{k(j)}}\|^{-1}\right)\right]\right|$$
$$\to 0,\quad \text{a.s.-}P_{Z^\infty}.$$

Since the argument started with an arbitrary subsequence n_k, equation (B.29) holds.

To conclude the proof of equation (B.26), we now show that equation (B.27) in fact holds (i.e., $T_n=o(1)$). Again, it suffices to show that for any subsequence, there exists a sub-sub-sequence such that $T_{n(j)}=o(1)$. For any subsequence, since $(\|u^*_n\|)_n$ is a bounded sequence (under Assumption 3.1(iv)), there exists a further subsubsequence (which we denote as $(n(j))_j$) such that $\lim_{j\to\infty}\|u^*_{n(j)}\|=d_\infty\in[c,C]$ for finite $c,C>0$. Observe that

$$\begin{aligned}T_{n(j)}\le{}&\sup_{f\in BL_1}\left|E\left[f\left(\left[\frac{\mathbb{Z}}{\|u^*_{n(j)}\|}\right]^2\right)\right]-E\left[f\left(\left[\frac{\mathbb{Z}}{d_\infty}\right]^2\right)\right]\right|\\&+\sup_{f\in BL_1}\left|E\left[f\left(\left[\frac{\mathbb{Z}}{d_\infty}\right]^2\right)\right]-E\left[f\left(\left(\frac{\|u^*_{n(j)}\|}{d_\infty}\right)^2\widehat{\text{QLR}}_{n(j)}(\phi_0)\right)\right]\right|\\&+\sup_{f\in BL_1}\left|E\left[f\left(\widehat{\text{QLR}}_{n(j)}(\phi_0)\right)\right]\right.\\&\left.-E\left[f\left(\left(\frac{\|u^*_{n(j)}\|}{d_\infty}\right)^2\widehat{\text{QLR}}_{n(j)}(\phi_0)\right)\right]\right|.\end{aligned}$$

The first term vanishes because $\mathbb{Z}$ is bounded in probability and $\lim_{j\to\infty}\|u^*_{n(j)}\|=d_\infty>0$; the third term follows by the same reason (by Theorem 4.3 and Assumption 3.6(ii), $\widehat{\text{QLR}}_n(\phi_0)$ is bounded in probability).

Finally, for any $f \in BL_1$, let $f(d_\infty^{-1}\cdot) \equiv f \circ d_\infty^{-2}(\cdot)$. Since $f \circ d_\infty^{-2}$ is bounded and $|f \circ d_\infty^{-2}(t) - f \circ d_\infty^{-2}(s)| \leq d_\infty^{-2}|t-s| \leq c^{-2}|t-s|$, we have $\{f \circ d_\infty^{-2} : f \in BL_1\} \subseteq BL_{c^{-2}}$. Therefore, the second term in the previous display is majorized by $\sup_{f \in BL_{c^{-2}}} |E[f([\mathbb{Z}]^2)] - E[f(\|u^*_{n(j)}\|^2 \times \widehat{\mathrm{QLR}}_{n(j)}(\phi_0))]|$. Hence, to conclude the proof we need to show that

$$\text{(B.30)} \qquad \lim_{j\to\infty} \sup_{f \in BL_{c^{-2}}} \left|E\left[f\left(\mathbb{Z}^2\right)\right] - E\left[f\left(\left\|u^*_{n(j)}\right\|^2 \times \widehat{\mathrm{QLR}}_{n(j)}(\phi_0)\right)\right]\right| = 0.$$

Theorem 4.3 (i.e., $\|u^*_n\|^2 \times \widehat{\mathrm{QLR}}_n(\phi_0) = [\sqrt{n}\mathbb{Z}_n]^2 + o_P(1)$) and Assumption 3.6(ii) directly imply that the above equation (B.30) actually holds for the whole sequence, which readily implies that for any subsequence $(n(j))_j$, there is a subsubsequence (which we still denote as $(n(j))_j$) for which the previous display holds.

Finally for *Result (2)*, we want to show that

$$\sup_{t\in\mathbb{R}} \left| P_{V^\infty|Z^\infty}\left(\frac{\widehat{\mathrm{QLR}}{}^B_n(\widehat{\phi}_n)}{\sigma^2_\omega} \leq t \,\middle|\, Z^n\right) - P_{Z^\infty}\left(\widehat{\mathrm{QLR}}_n(\phi_0) \leq t | H_0\right)\right|$$
$$= o_{P_{Z^\infty}}(1).$$

Let $f_t(\cdot) \equiv 1\{\cdot \leq t\}$ for $t \in \mathbb{R}$. Under this notation, the previous display can be cast as

$$A_n \equiv \sup_{t\in\mathbb{R}} \left| E_{P_{V^\infty|Z^\infty}}\left[f_t\left(\frac{\widehat{\mathrm{QLR}}{}^B_n(\widehat{\phi}_n)}{\sigma^2_\omega}\right) \middle| Z^n\right] - E_{P_{Z^\infty}}\left[f_t\left(\widehat{\mathrm{QLR}}_n(\phi_0)\right)\right]\right|$$
$$= o_{P_{Z^\infty}}(1).$$

Denote $\mathbb{Z}^2 \sim \chi^2_1$ and

$$A_{1,n} \equiv \sup_{t'\in\mathbb{R}} \left| E_{P_{V^\infty|Z^\infty}}\left[f_{t'}\left(\left\|u^*_n\right\|^2 \times \frac{\widehat{\mathrm{QLR}}{}^B_n(\widehat{\phi}_n)}{\sigma^2_\omega}\right) \middle| Z^n\right] - E\left[f_{t'}\left(\mathbb{Z}^2\right)\right]\right|,$$
$$A_{2,n} \equiv \sup_{t'\in\mathbb{R}} \left| E_{P_{Z^\infty}}\left[f_{t'}\left(\left\|u^*_n\right\|^2 \times \widehat{\mathrm{QLR}}_n(\phi_0)\right)\right] - E\left[f_{t'}\left(\mathbb{Z}^2\right)\right]\right|.$$

Notice that

$$A_n = \sup_{t\in\mathbb{R}} \left| E_{P_{V^\infty|Z^\infty}}\left[f_{t\|u^*_n\|^2}\left(\left\|u^*_n\right\|^2 \frac{\widehat{\mathrm{QLR}}{}^B_n(\widehat{\phi}_n)}{\sigma^2_\omega}\right) \middle| Z^n\right]\right.$$
$$\left. - E_{P_{Z^\infty}}\left[f_{t\|u^*_n\|^2}\left(\left\|u^*_n\right\|^2 \widehat{\mathrm{QLR}}_n(\phi_0)\right)\right]\right|$$

$$
\begin{aligned}
&\le \sup_{t\in\mathbb{R}} \sup_{d\in[c,C]} \Bigg| E_{P_{V^\infty|Z^\infty}}\Bigg[f_{td^2}\Bigg(\|u_n^*\|^2 \frac{\widehat{\mathrm{QLR}}{}_n^B(\widehat{\phi}_n)}{\sigma_\omega^2}\Bigg) \Bigg| Z^n \Bigg] \\
&\quad - E_{P_{Z^\infty}}\big[f_{td^2}\big(\|u_n^*\|^2 \widehat{\mathrm{QLR}}_n(\phi_0)\big)\big] \Bigg| \\
&\le \sup_{t'\in\mathbb{R}} \Bigg| E_{P_{V^\infty|Z^\infty}}\Bigg[f_{t'}\Bigg(\|u_n^*\|^2 \times \frac{\widehat{\mathrm{QLR}}{}_n^B(\widehat{\phi}_n)}{\sigma_\omega^2}\Bigg) \Bigg| Z^n \Bigg] \\
&\quad - E_{P_{Z^\infty}}\big[f_{t'}\big(\|u_n^*\|^2 \times \widehat{\mathrm{QLR}}_n(\phi_0)\big)\big] \Bigg| \\
&\le A_{1,n} + A_{2,n}
\end{aligned}
$$

where the first line follows from the property that $f_t(\cdot) = f_{t\lambda}(\lambda \times \cdot)$ for any $\lambda \in \mathbb{R}_+$; the second line follows because by assumption, $\|u_n^*\|^2 \in [c^2, C^2]$; the third line follows simply because $\{1\{\cdot \le t\lambda\} : t \in \mathbb{R} \text{ and } \lambda \in \mathbb{R}_+\} \subseteq \{1\{\cdot \le t\} : t \in \mathbb{R}\}$. Finally, the last line is due to the triangle inequality and the definitions of $A_{1,n}$ and $A_{2,n}$.

By Theorem 4.3, under the null, $\|u_n^*\|^2 \times \widehat{\mathrm{QLR}}_n(\phi_0)$ converges weakly to $\mathbb{Z}^2 \sim \chi_1^2$, whose distribution is continuous. Therefore, by Polya's theorem, $A_{2,n} = o(1)$. Similarly,

$$
\begin{aligned}
A_{1,n} &= \sup_{t'\in\mathbb{R}} \Bigg| P_{V^\infty|Z^\infty}\Bigg(\|u_n^*\|^2 \times \frac{\widehat{\mathrm{QLR}}{}_n^B(\widehat{\phi}_n)}{\sigma_\omega^2} \le t' \Bigg| Z^n \Bigg) - P\big(\mathbb{Z}^2 \le t'\big) \Bigg| \\
&= o_{P_{Z^\infty}}(1)
\end{aligned}
$$

by equation (B.26) and by the same arguments in Lemma 10.11 in Kosorok (2008). *Q.E.D.*

We first recall some notation introduced in the main text. Let $\mathcal{T}_n \equiv \{t \in \mathbb{R} : |t| \le 4M_n^2\delta_n\}$. For $t_n \in \mathcal{T}_n$, $\alpha(t_n) \equiv \alpha + t_n u_n^*$ where $u_n^* = v_n^*/\|v_n^*\|_{\mathrm{sd}}$ and $v_n^* = (v_{\theta,n}^{*\prime}, v_{h,n}^*(\cdot))'$. To simplify presentation, we use $r_n = r_n(t_n) \equiv (\max\{t_n^2, t_n n^{-1/2}, o(n^{-1})\})^{-1}$.

PROOF OF LEMMA 5.1: For *Result (1)*, if $\omega \equiv 1$, then Assumption Boot.3(i) simplifies to

$$
\begin{aligned}
&P_{Z^\infty}\Bigg(P_{V^\infty|Z^\infty}\Bigg(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n \Bigg| \widehat{\Lambda}_n\big(\alpha(t_n),\alpha\big) \\
&\quad - t_n\big\{\mathbb{Z}_n + \langle u_n^*, \alpha - \alpha_0\rangle\big\} - \frac{B_n}{2} t_n^2 \Bigg| \ge \delta \Bigg| Z^n \Bigg) \le \delta \Bigg) \ge 1 - \delta;
\end{aligned}
$$

iff

$$P_{Z^\infty}\left(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n\left|\widehat{\Lambda}_n\big(\alpha(t_n),\alpha\big)-t_n\big\{\mathbb{Z}_n+\langle u_n^*,\alpha-\alpha_0\rangle\big\}-\frac{B_n}{2}t_n^2\right|\le\delta\right)$$
$$\ge 1-\delta,$$

where $\widehat{\Lambda}_n(\alpha(t_n),\alpha)\equiv 0.5(\widehat{Q}_n(\alpha(t_n))-\widehat{Q}_n(\alpha))$ and B_n is a Z^n measurable random variable with $B_n=O_{P_{Z^\infty}}(1)$. Therefore, if we could verify Assumption Boot.3(i) in Result (2), we also verify Assumption 3.6(i).

For *Result (2)*, we divide its proof in several steps.

STEP 1: We first introduce some notation. Let

$$P_n\big(Z^n\big)\equiv P_{V^\infty|Z^\infty}\left(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n\left|\widehat{\Lambda}_n^B\big(\alpha(t_n),\alpha\big)\right.\right.$$
$$\left.\left.-t_n\big\{\mathbb{Z}_n^\omega+\langle u_n^*,\alpha-\alpha_0\rangle\big\}-\frac{B_n^\omega}{2}t_n^2\right|\ge\delta\Big|Z^n\right).$$

Recall that $\ell_n^B(x,\alpha)\equiv\widetilde{m}(x,\alpha)+\widehat{m}^B(x,\alpha_0)$. Let

$$\widehat{L}_n^B\big(\alpha(t_n),\alpha\big)\equiv\frac{1}{2n}\sum_{i=1}^n\big\{\ell_n^B\big(X_i,\alpha(t_n)\big)'\widehat{\Sigma}(X_i)^{-1}\ell_n^B\big(X_i,\alpha(t_n)\big)$$
$$-\ell_n^B(X_i,\alpha)'\widehat{\Sigma}(X_i)^{-1}\ell_n^B(X_i,\alpha)\big\}.$$

We need to show that $P_{Z^\infty}(P_n(Z^n)<\delta)\ge 1-\delta$ eventually, which is equivalent to showing that $P_{Z^\infty}(P_n(Z^n)>\delta)\le\delta$ eventually. Hence, it suffices to show that

$$P_{Z^\infty}\big(\big\{P_n'\big(Z^n\big)>\delta\big\}\cap S_n\big)+P_{Z^\infty}\big(S_n^C\big)\le\delta,\quad\text{eventually,}$$

for some event S_n that is measurable with respect to Z^n, and some $P_n'(Z^n)\ge P_n(Z^n)$ a.s.; here S_n^C denotes the complement of S_n. In the following, we take

$$S_n\equiv\left\{Z^n:P_{V^\infty|Z^\infty}\Big(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n\big|\widehat{\Lambda}_n^B\big(\alpha(t_n),\alpha\big)-\widehat{L}_n^B\big(\alpha(t_n),\alpha\big)\big|\right.$$
$$\left.\ge 0.5\delta\big|Z^n\Big)<0.5\delta\right\},$$

and

$$P_n'\big(Z^n\big)\equiv P_{V^\infty|Z^\infty}\left(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n\left|\widehat{L}_n^B\big(\alpha(t_n),\alpha\big)\right.\right.$$

$$- t_n\{\mathbb{Z}_n^\omega + \langle u_n^*, \alpha - \alpha_0\rangle\} - \frac{B_n^\omega}{2} t_n^2 \Big| \geq 0.5\delta \Big| Z^n\Big)$$

$$+ P_{V^\infty|Z^\infty}\Big(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n \big|\widehat{\Lambda}_n^B(\alpha(t_n),\alpha) - \widehat{L}_n^B(\alpha(t_n),\alpha)\big|$$

$$\geq 0.5\delta \big| Z^n\Big).$$

It follows that we "only" need to show that

$$P_{Z^\infty}(S_n^C) \leq 0.5\delta \quad \text{and}$$
$$P_{Z^\infty}(\{P_n'(Z^n) > \delta\} \cap S_n) \leq 0.5\delta, \quad \text{eventually.}$$

Since $P_{Z^\infty}(S_n^C)$ can be expressed as

$$P_{Z^\infty}\Big(P_{V^\infty|Z^\infty}\Big(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n \big|\widehat{\Lambda}_n^B(\alpha(t_n),\alpha) - \widehat{L}_n^B(\alpha(t_n),\alpha)\big| \geq 0.5\delta \big| Z^n\Big)$$
$$\geq 0.5\delta\Big),$$

which, by Lemma A.2(3), is in fact less than 0.5δ, we only need to verify

$$P_{Z^\infty}(\{P_n'(Z^n) > \delta\} \cap S_n) \leq 0.5\delta, \quad \text{eventually.}$$

It is easy to see that

$$P_{Z^\infty}(\{P_n'(Z^n) > \delta\} \cap S_n)$$
$$\leq P_{Z^\infty}\Big(P_{V^\infty|Z^\infty}\Big(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n \Big|\widehat{L}_n^B(\alpha(t_n),\alpha)$$
$$- t_n\{\mathbb{Z}_n^\omega + \langle u_n^*, \alpha - \alpha_0\rangle\} - \frac{B_n^\omega}{2} t_n^2 \Big| \geq 0.5\delta \Big| Z^n\Big) > 0.5\delta\Big).$$

Hence, in order to prove the desired result, it suffices to show that

$$\text{(B.31)} \quad P_{Z^\infty}\Big(P_{V^\infty|Z^\infty}\Big(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n \Big|\widehat{L}_n^B(\alpha(t_n),\alpha)$$
$$- t_n\{\mathbb{Z}_n^\omega + \langle u_n^*, \alpha - \alpha_0\rangle\} - \frac{B_n^\omega}{2} t_n^2 \Big| \geq \delta \Big| Z^n\Big) > \delta\Big) < \delta$$

eventually.

STEP 2: For any $\alpha \in \mathcal{N}_{osn}$ and $t_n \in \mathcal{T}_n$, $\alpha(t_n) = \alpha + t_n u_n^*$, under Assumption A.7(i), we can apply the mean value theorem (w.r.t. t_n) and obtain

$$\begin{aligned}
\widehat{L}_n^B\big(\alpha(t_n), \alpha\big) &= \frac{t_n}{n}\sum_{i=1}^{n}\left(\frac{d\widetilde{m}(X_i,\alpha)}{d\alpha}[u_n^*]\right)' \widehat{\Sigma}(X_i)^{-1}\ell_n^B(X_i,\alpha) \\
&\quad + \frac{t_n^2}{2n}\int_0^1 \sum_{i=1}^{n}\left(\frac{d\widetilde{m}\big(X_i,\alpha(s)\big)}{d\alpha}[u_n^*]\right)' \\
&\quad \times \widehat{\Sigma}(X_i)^{-1}\left(\frac{d\widetilde{m}\big(x,\alpha(s)\big)}{d\alpha}[u_n^*]\right) ds \\
&\quad + \frac{t_n^2}{2n}\int_0^1 \sum_{i=1}^{n}\left(\frac{d^2\widetilde{m}\big(X_i,\alpha(s)\big)}{d\alpha^2}[u_n^*, u_n^*]\right)' \\
&\quad \times \widehat{\Sigma}(X_i)^{-1}\ell_n^B\big(Z_i,\alpha(s)\big)\, ds \\
&\equiv t_n T_{1n}^B(\alpha) + \frac{t_n^2}{2}\big\{T_{2n}(\alpha) + T_{3n}^B(\alpha)\big\},
\end{aligned}$$

where $\alpha(s) \equiv \alpha + s t_n u_n^* \in \mathcal{N}_{osn}$.

From these calculations and the fact that $P_{V^\infty|Z^\infty}(a_n + b_n \geq d|Z^n) \leq P_{V^\infty|Z^\infty}(a_n \geq 0.5d|Z^n) + P_{V^\infty|Z^\infty}(b_n \geq 0.5d|Z^n)$ a.s. for any two measurable random variables a_n and b_n, it follows that

$$\begin{aligned}
&P_{V^\infty|Z^\infty}\left(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n\left|\widehat{L}_n^B\big(\alpha(t_n),\alpha\big) - t_n\big\{\mathbb{Z}_n^\omega + \langle u_n^*, \alpha-\alpha_0\rangle\big\} - \frac{B_n^\omega}{2}t_n^2\right|\right. \\
&\qquad \left. \geq 0.5\delta \,\middle|\, Z^n\right) \\
&\quad \leq P_{V^\infty|Z^\infty}\Big(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n t_n\big|T_{1n}^B(\alpha) - \big\{\mathbb{Z}_n^\omega + \langle u_n^*, \alpha-\alpha_0\rangle\big\}\big| \\
&\qquad \geq 0.25\delta\big|Z^n\Big) \\
&\qquad + P_{V^\infty|Z^\infty}\left(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n\frac{t_n^2}{2}\big|\big\{T_{2n}(\alpha) + T_{3n}^B(\alpha)\big\} - B_n^\omega\big|\right. \\
&\qquad \left. \geq 0.25\delta\,\middle|\, Z^n\right).
\end{aligned}$$

Hence, in order to show equation (B.31), it suffices to show that

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n} r_n t_n\left|T^B_{1n}(\alpha)-\left\{\mathbb{Z}^\omega_n+\langle u^*_n,\alpha-\alpha_0\rangle\right\}\right|\geq\delta\middle|Z^n\right)\geq\delta\right)<\delta$$

and

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sup_{(\alpha,t_n)\in\mathcal{N}_{osn}\times\mathcal{T}_n}\frac{r_n t_n^2}{2}\left|\left\{T_{2n}(\alpha)+T^B_{3n}(\alpha)\right\}-B^\omega_n\right|\geq\delta\middle|Z^n\right)\geq\delta\right)<\delta$$

eventually.

Since $r_n t_n \leq n^{1/2}$, by Lemma A.3, the first equation holds. Since $r_n t_n^2 \leq 1$, in order to verify the second equation it suffices to verify that, for any $\delta > 0$,

$$P_{Z^\infty}\left(\sup_{\alpha\in\mathcal{N}_{osn}}\left|T_{2n}(\alpha)-B^\omega_n\right|\geq\delta\right)<\delta,\quad \forall n\geq N(\delta),$$

and

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sup_{\alpha\in\mathcal{N}_{osn}}\left|T^B_{3n}(\alpha)\right|\geq\delta\middle|Z^n\right)\geq\delta\right)<\delta,\quad \forall n\geq N(\delta).$$

By Lemmas A.5(1) and A.4, these two equations hold.

By our choice of $\ell^B_n()$ (in particular, the fact that $\widetilde{m}$ is measurable with respect to Z^n), it follows that $B^\omega_n = B_n = O_{P_{V^\infty|Z^\infty}}(1)$ wpa1 (P_{Z^∞}). Thus we verified Assumption Boot.3(i).

Finally, Lemma A.5(2) implies $|B^\omega_n - \|u^*_n\|^2| = o_{P_{V^\infty|Z^\infty}}(1)$ wpa1 (P_{Z^∞}) and $|B_n - \|u^*_n\|^2| = o_{P_{Z^\infty}}(1)$. *Q.E.D.*

The following lemma is a LLN for triangular arrays.

LEMMA B.4: *Let* $((X_{i,n})^n_{i=1})^\infty_{n=1}$ *be a triangular array of real-valued random variables such that* (a) $X_{1,n},\dots,X_{n,n}$ *are independent and* $X_{i,n}\sim P_{i,n}$, *for all* n, (b) $E[X_{i,n}]=0$ *for all* i *and* n, *and* (c) *there is a sequence of nonnegative real numbers* $(b_n)_n$ *such that* $b_n=o(\sqrt{n})$ *and*

$$\limsup_{n\to\infty} n^{-1}\sum_{i=1}^n E\left[|X_{i,n}|1\left\{|X_{i,n}|\geq b_n\right\}\right]=0.$$

Then: *for all* $\epsilon > 0$, *there is a* $N(\epsilon)$ *such that*

$$\Pr\left(\left|n^{-1}\sum_{i=1}^{n} X_{i,n}\right| \geq \epsilon\right) < \epsilon \quad \textit{for all } n \geq N(\epsilon).$$

PROOF: We obtain the result by modifying the proofs of Billingsley (1995, Theorem 22.1) and of Feller (1970, p. 248). For any $\epsilon > 0$, let

$$X_{i,n} = X_{i,n}1\{|X_{i,n}| \leq b_n\} + X_{i,n}1\{|X_{i,n}| > b_n\} \equiv X_{i,n}^B + X_{i,n}^U.$$

Thus,

$$\begin{aligned}
&\Pr\left(\left|n^{-1}\sum_{i=1}^{n} X_{i,n}\right| \geq \epsilon\right) \\
&\quad \leq \Pr\left(\left|n^{-1}\sum_{i=1}^{n} X_{i,n}^B\right| \geq 0.5\epsilon\right) + \Pr\left(\left|n^{-1}\sum_{i=1}^{n} X_{i,n}^U\right| \geq 0.5\epsilon\right) \\
&\quad \equiv T_{1,\epsilon} + T_{2,\epsilon}.
\end{aligned}$$

By conditions (b) and (c), it is easy to see that, for large enough n,

$$\begin{aligned}
T_{1,\epsilon} &\leq \Pr\left(\left|n^{-1}\sum_{i=1}^{n}\{X_{i,n}^B - E[X_{i,n}^B]\}\right| \geq 0.25\epsilon\right) \\
&\qquad + 1\left\{n^{-1}\sum_{i=1}^{n} E[X_{i,n}^B] \geq 0.25\epsilon\right\} \\
&= \Pr\left(\left|n^{-1}\sum_{i=1}^{n}\{X_{i,n}^B - E[X_{i,n}^B]\}\right| \geq 0.25\epsilon\right) \leq 2\exp\left(-\text{const.}\frac{\epsilon^2 n}{b_n^2}\right),
\end{aligned}$$

for some finite constant const. > 0, where the last inequality is due to Hoeffding inequality (cf. Van der Vaart and Wellner (1996, Appendix A.6)). Thus, there is a $N(\epsilon)$ such that, for all $n \geq N(\epsilon)$, $T_{1,\epsilon} < 0.5\epsilon$.

For $T_{2,\epsilon}$, by Markov inequality and then by condition (c), we have

$$\begin{aligned}
T_{2,\epsilon} &\leq (\epsilon/2)^{-1} n^{-1}\sum_{i=1}^{n}\int_{\{|x|\geq b_n\}} |x| P_{i,n}(dx) \\
&= (\epsilon/2)^{-1} n^{-1}\sum_{i=1}^{n}\int |x| 1\{|x| \geq b_n\} P_{i,n}(dx) < 0.5\epsilon
\end{aligned}$$

eventually. *Q.E.D.*

PROOF OF LEMMA 5.2: We divide the proof into several steps.

STEP 1: We first show that the event

$$S_n \equiv \Big\{ Z^n : \Big| n^{-1} \sum_{i=1}^{n} \big(g\big(X_i, u_n^*\big)\rho(Z_i, \alpha_0)\big)^2 - E\big[g\big(X, u_n^*\big)\Sigma_0(X) g\big(X, u_n^*\big)'\big] \Big| \leq \delta \Big\}$$

*occurs wpa*1 (P_{Z^∞}). For this, we apply Lemma B.4. Using the notation in the lemma, we let $X_{i,n} \equiv (g(X_i, u_n^*)\rho(Z_i, \alpha_0))^2 - E[g(X, u_n^*)\Sigma_0(X)g(X, u_n^*)']$, and thus conditions (a) and (b) of Lemma B.4 immediately follow (note that $E[g(X, u_n^*)\Sigma_0(X)g(X, u_n^*)'] = 1$). In order to check condition (c), note first that for any generic random variable X with mean $\mu < \infty$, it follows that

$$\begin{aligned} &E\big[|X-\mu| 1\big\{|X-\mu| \geq b_n\big\}\big] \\ &\quad \leq E\big[|X| 1\big\{|X| \geq b_n - |\mu|\big\}\big] + |\mu| \Pr\big\{|X| \geq b_n - |\mu|\big\}. \end{aligned}$$

Since b_n is taken to diverge, we can "redefine" b_n as $b_n - |\mu|$. Moreover,

$$\Pr\big\{|X| \geq b_n - |\mu|\big\} \leq E\big[\max\big\{|X|, 1\big\} 1\big\{|X| \geq b_n - |\mu|\big\}\big].$$

Again, since b_n is taken to diverge, the only relevant case is $|X| \geq 1$. Therefore, it suffices to study $E[|X| 1\{|X| \geq b_n\}]$ in order to bound $E[|X-\mu| 1\{|X-\mu| \geq b_n\}]$. Thus, applied to our case, it is sufficient to verify that

$$\begin{aligned} &\limsup_{n\to\infty} n^{-1} \sum_{i=1}^{n} E\big[\big(g\big(X_i, u_n^*\big)\rho(Z_i, \alpha_0)\big)^2 \\ &\quad \times 1\big\{\big(g\big(X_i, u_n^*\big)\rho(Z_i, \alpha_0)\big)^2 \geq b_n\big\}\big] = 0, \end{aligned}$$

which holds under our assumption equation (5.1).

STEP 2: Let $\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \zeta_i \mathbf{s}_{i,n}$, where $\mathbf{s}_{i,n} \equiv g(X_i, u_n^*)\rho(Z_i, \alpha_0)$, and either$\{\zeta_i\}_{i=1}^n$ is i.i.d. with $\zeta_i = (\omega_i - 1)\sigma_\omega^{-1}$ (under Assumption Boot.1) or$\{\zeta_i\}_{i=1}^n$ is multinomial with $\zeta_i = (\omega_{i,n} - 1)$ (under Assumption Boot.2). In the following, we let P_Ω denote the conditional distribution of $\{\zeta_i\}_{i=1}^n$ given the data Z^n, which is also the unconditional distribution of $\{\zeta_i\}_{i=1}^n$ since $\{\zeta_i\}_{i=1}^n$ is independent of Z^n. We want to establish that

$$\sup_{f \in BL_1} \left| E\left[f\left(\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega}\right) \Big| Z^n \right] - E\big[f(\mathbb{Z})\big] \right| = o_{P_{Z^\infty}}(1),$$

where $\mathbb{Z} \sim N(0,1)$. Which is equivalent to showing that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\zeta_i \mathbf{s}_{i,n} \Rightarrow \mathbb{Z}, \quad \text{wpa1 } (P_{Z^\infty}).$$

Which, by Billingsley (1995, Theorem 20.5, p. 268), in turn suffices to show that any subsequence contains a further subsequence, $(n_k)_k$, such that

$$\frac{1}{\sqrt{n_k}}\sum_{i=1}^{n_k}\zeta_i \mathbf{s}_{i,n_k} \Rightarrow \mathbb{Z}, \quad \text{a.s.-}(P_{Z^\infty}). \tag{B.32}$$

Step 3 below establishes (B.32) under Assumption Boot.1, while Step 4 below establishes (B.32) under Assumption Boot.2.

STEP 3—Under Assumption Boot.1: Since the event S_n occurs wpa1 (P_{Z^∞}) (Step 1), it follows that any subsequence contains a further subsequence such that $n_k^{-1}\sum_{i=1}^{n_k}(\mathbf{s}_{i,n_k})^2 \to 1$, a.s.-$(P_{Z^\infty})$. Moreover, $\max_{i\leq n_k}|\mathbf{s}_{i,n_k}|/\sqrt{n_k} = o(1)$, a.s.-$(P_{Z^\infty})$. This follows since, for any $\epsilon > 0$,

$$\begin{aligned} P_{Z^\infty}\Big(\max_{i\leq n}|\mathbf{s}_{i,n}| \geq \epsilon\sqrt{n}\Big) &\leq \sum_{i=1}^{n}\int_{|s|\geq\epsilon\sqrt{n}} P_{i,n}(ds) \\ &\leq \epsilon^{-2}n^{-1}\sum_{i=1}^{n}\int_{|s|\geq\epsilon\sqrt{n}} s^2 P_{i,n}(ds) \\ &= \epsilon^{-2}n^{-1}\sum_{i=1}^{n} E\big[\mathbf{s}_{i,n}^2 1\{|\mathbf{s}_{i,n}| \geq \epsilon\sqrt{n}\}\big]. \end{aligned}$$

We note that $1\{|\mathbf{s}_{i,n}| \geq \epsilon\sqrt{n}\} \leq 1\{|\mathbf{s}_{i,n}|^2 \geq b_n\}$ (provided that $|\mathbf{s}_{i,n}| \geq 1$, but if it is not, then the proof is trivial). Hence by equation (5.1) and the fact that $\mathbf{s}_{i,n}$ are row-wise i.i.d., the RHS is of order $o(1)$. Going to a subsequence establishes the result.

Under Assumption Boot.1, $\zeta_i = (\omega_i - 1)\sigma_\omega^{-1}$ is i.i.d. with mean zero, variance 1, hence conditional on the event S_n, for any $\epsilon > 0$,

$$\begin{aligned} &n_k^{-1}\sum_{i=1}^{n_k} E_{P_\Omega}\big[(\zeta_i \mathbf{s}_{i,n_k})^2 1\{|\zeta_i \mathbf{s}_{i,n_k}| > \epsilon\sqrt{n_k}\}\big] \\ &\leq \left(n_k^{-1}\sum_{i=1}^{n_k}|\mathbf{s}_{i,n_k}|^2\right) \times E_{P_\Omega}\Big[\zeta_1^2 \times 1\Big\{|\zeta_1| \times \max_{1\leq i\leq n}|\mathbf{s}_{i,n_k}| > \epsilon\sqrt{n_k}\Big\}\Big] \\ &\leq \left(n_k^{-1}\sum_{i=1}^{n_k}|\mathbf{s}_{i,n_k}|^2\right) \times E_{P_\Omega}\big[\zeta_1^2 \times 1\{|\zeta_1| > \epsilon/\epsilon'\}\big] \to 0, \quad \text{a.s.-}(P_{Z^\infty}), \end{aligned}$$

where the second inequality follows from the fact that $\max_{i\leq n_k}|\mathbf{s}_{i,n_k}|/\sqrt{n_k}<\epsilon'$, a.s.-$(P_{Z^\infty})$ eventually. Since ζ_1 are i.i.d., by choosing the ϵ' (small relative to ϵ), one can make the term $E_{P_\Omega}[\zeta_1^2 1\{|\zeta_1|>\epsilon/\epsilon'\}]$ arbitrarily small. The Lindeberg–Feller CLT then implies that $\frac{1}{\sqrt{n_k}}\sum_{i=1}^{n_k}\zeta_i\mathbf{s}_{i,n_k}\Rightarrow\mathbb{Z}$, a.s.-$(P_{Z^\infty})$ where $\mathbb{Z}\sim N(0,1)$.

We have thus showed that any subsequence contains a further subsequence such that the above equation holds; therefore,

$$\sup_{f\in BL_1}\left|E\left[f\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\zeta_i\mathbf{s}_{i,n}\right)\Big|Z^n\right]-E\big[f(\mathbb{Z})\big]\right|=o_{P_{Z^\infty}}(1).$$

STEP 4—Under Assumption Boot.2: We proceed as in Step 3 to establish equation (B.32). The difference is that now $\{\zeta_i\}_{i=1}^n$ is not i.i.d., but exchangeable with $\zeta_i=\zeta_{i,n}\equiv(\omega_{i,n}-1)$. To overcome this, we follow Lemma 3.6.15 (or really Proposition A.5.3) in VdV-W for a given subsequence $(n_k)_k$. To simplify notation, we let $n=n_k$ and $\mathbf{s}_{i,n}=\mathbf{s}_{i,n_k}$.

Under Assumption Boot.2 we have: $n^{-1}\sum_{i=1}^n\zeta_{i,n}=0$, $n^{-1}\sum_{i=1}^n\zeta_{i,n}^2\to1$, $n^{-1}\max_{1\leq i\leq n}\zeta_{i,n}^2=o_{P_\Omega}(1)$, and $\max_{1\leq i\leq n}E[\zeta_{i,n}^4]\leq c<\infty$. Conditional on the event S_n, we also have $n^{-1}\sum_{i=1}^n\mathbf{s}_{i,n}\to0$, $n^{-1}\sum_{i=1}^n\mathbf{s}_{i,n}^2\to1$, and $n^{-1}\times\max_{1\leq i\leq n}\mathbf{s}_{i,n}^2=o(1)$ (this has already been established in Step 3), and finally we need:

$$\limsup_{n\to\infty}n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathbf{s}_{i,n}\zeta_{j,n})^2 1\big\{|\mathbf{s}_{i,n}\zeta_{j,n}|>\epsilon\sqrt{n}\big\}=0,\quad \text{a.s.-}P_{Z^\infty}. \tag{B.33}$$

To show equation (B.33), we note that

$$\begin{aligned}&\limsup_{n\to\infty}n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathbf{s}_{i,n}\zeta_{j,n})^2 1\big\{|\mathbf{s}_{i,n}\zeta_{j,n}|>\epsilon\sqrt{n}\big\}\\&\quad\leq\limsup_{n\to\infty}\left(\frac{\sum_{i=1}^{n}(\zeta_{i,n})^2}{n}\times\frac{\sum_{j=1}^{n}(\mathbf{s}_{j,n})^2\times1\Big\{|\mathbf{s}_{j,n}|\times\max_{1\leq i\leq n}|\zeta_{i,n}|>\epsilon\sqrt{n}\Big\}}{n}\right).\end{aligned}$$

Under Assumption Boot.2, conditional on the event S_n we have, for any $\epsilon>0$,

$$\begin{aligned}&\limsup_{n->\infty}E_{P_\Omega}\left(\frac{\sum_{j=1}^{n}(\mathbf{s}_{j,n})^2\times1\Big\{|\mathbf{s}_{j,n}|\times\max_{1\leq i\leq n}|\zeta_{i,n}|>\epsilon\sqrt{n}\Big\}}{n}\right)\\&\quad=0,\quad\text{by equation (5.1).}\end{aligned}$$

Hence (with possibly going to a subsequence) we establish equation (B.33). So, by Lemma 3.6.15 (or Proposition A.5.3) in Van der Vaart and Wellner (1996),

$$\frac{1}{\sqrt{n_k}}\sum_{i=1}^{n_k}\zeta_i \mathbf{s}_{i,n_k} \Rightarrow \mathbb{Z}, \quad \text{a.s.-}(P_{Z^\infty}).$$

The rest of the steps are analogous to those in Step 3 and will not be repeated here. *Q.E.D.*

B.3.1. *Alternative Bootstrap Sieve t Statistics*

In this subsection, we present additional bootstrap sieve t statistics. Recall that $\widehat{W}_n \equiv \sqrt{n}\frac{\phi(\widehat{\alpha}_n)-\phi(\alpha_0)}{\|\widehat{v}_n^*\|_{n,\mathrm{sd}}}$ is the original-sample sieve t statistic. The first one is $\widehat{W}_{1,n}^B \equiv \sqrt{n}\frac{\phi(\widehat{\alpha}_n^B)-\phi(\widehat{\alpha}_n)}{\sigma_\omega\|\widehat{v}_n^*\|_{n,\mathrm{sd}}}$. In the definition of $\widehat{W}_{2,n}^B$ one could also define $\|\widehat{v}_n^*\|_{B,\mathrm{sd}}^2$ using $\widehat{\Sigma}_{0i}^B = \widehat{E}_n[\varrho(V,\widehat{\alpha}_n)\varrho(V,\widehat{\alpha}_n)'|X=X_i]$ instead of $\varrho(V_i,\widehat{\alpha}_n)\varrho(V_i,\widehat{\alpha}_n)'$, which will be a bootstrap analog to $\|\widehat{v}_n^*\|_{n,\mathrm{sd}}^2$ defined in equation (B.5).

Let $\widehat{W}_{3,n}^B \equiv \sqrt{n}\frac{\phi(\widehat{\alpha}_n^B)-\phi(\widehat{\alpha}_n)}{\|\widehat{v}_n^B\|_{B,\mathrm{sd}}}$, where $\|\widehat{v}_n^B\|_{B,\mathrm{sd}}^2$ is a bootstrap sieve variance estimator that is constructed as follows. First, we define

$$\|\cdot\|_{B,M}^2 \equiv n^{-1}\sum_{i=1}^n\left(\frac{d\widehat{m}^B(X_i,\widehat{\alpha}_n^B)}{d\alpha}[\cdot]\right)' M_{n,i}\left(\frac{d\widehat{m}^B(X_i,\widehat{\alpha}_n^B)}{d\alpha}[\cdot]\right),$$

where $M_{n,i}$ is some (almost surely) positive definite weighting matrix. Let $\widehat{v}_n^B$ be a *bootstrapped empirical Riesz representer* of the linear functional $\frac{d\phi(\widehat{\alpha}_n^B)}{d\alpha}[\cdot]$ under $\|\cdot\|_{B,\widehat{\Sigma}^{-1}}$. We compute a bootstrap sieve variance estimator as

$$\text{(B.34)} \quad \|\widehat{v}_n^B\|_{B,\mathrm{sd}}^2 \equiv \frac{1}{n}\sum_{i=1}^n\left(\frac{d\widehat{m}^B(X_i,\widehat{\alpha}_n^B)}{d\alpha}[\widehat{v}_n^B]\right)'\widehat{\Sigma}_i^{-1}\varrho(V_i,\widehat{\alpha}_n^B)\varrho(V_i,\widehat{\alpha}_n^B)'\widehat{\Sigma}_i^{-1} \times\left(\frac{d\widehat{m}^B(X_i,\widehat{\alpha}_n^B)}{d\alpha}[\widehat{v}_n^B]\right)$$

with $\varrho(V_i,\alpha) \equiv (\omega_{i,n}-1)\rho(Z_i,\alpha) \equiv \rho^B(V_i,\alpha)-\rho(Z_i,\alpha)$ for any α. That is, $\|\widehat{v}_n^B\|_{B,\mathrm{sd}}^2$ is a bootstrap analog to $\|\widehat{v}_n^*\|_{n,\mathrm{sd}}^2$ defined in equation (4.7). One could also define $\|\widehat{v}_n^B\|_{B,\mathrm{sd}}^2$ using $\widehat{E}_n[\varrho(V,\widehat{\alpha}_n^B)\varrho(V,\widehat{\alpha}_n^B)'|X=X_i]$ instead of $\varrho(V_i,\widehat{\alpha}_n^B)\varrho(V_i,\widehat{\alpha}_n^B)'$, which will be a bootstrap analog to $\|\widehat{v}_n^*\|_{n,\mathrm{sd}}^2$ defined in equation (B.5). In addition, one could also define $\|\widehat{v}_n^B\|_{B,\mathrm{sd}}^2$ using $\widehat{\alpha}_n$ instead of $\widehat{\alpha}_n^B$. In terms of the first order asymptotic approximation, this alternative definition yields the same asymptotic results. Due to space considerations, we omit these alternative bootstrap sieve variance estimators.

The bootstrap sieve variance estimator $\|\widehat{v}_n^B\|_{B,\mathrm{sd}}^2$ also has a closed form expression: $\|\widehat{v}_n^B\|_{B,\mathrm{sd}}^2 = (\widehat{F}_n^B)'(\widehat{D}_n^B)^{-1}\widehat{\mho}_{3,n}^B(\widehat{D}_n^B)^{-1}\widehat{F}_n^B$ with

$$\widehat{F}_n^B = \frac{d\phi(\widehat{\alpha}_n^B)}{d\alpha}\big[\overline{\psi}^{k(n)}(\cdot)'\big],$$

$$\widehat{D}_n^B = \frac{1}{n}\sum_{i=1}^n \left(\frac{d\widehat{m}^B(X_i,\widehat{\alpha}_n^B)}{d\alpha}\big[\overline{\psi}^{k(n)}(\cdot)'\big]\right)'\widehat{\Sigma}_i^{-1}\left(\frac{d\widehat{m}^B(X_i,\widehat{\alpha}_n^B)}{d\alpha}\big[\overline{\psi}^{k(n)}(\cdot)'\big]\right),$$

$$\widehat{\mho}_{3,n}^B = \frac{1}{n}\sum_{i=1}^n \left(\frac{d\widehat{m}^B(X_i,\widehat{\alpha}_n^B)}{d\alpha}\big[\overline{\psi}^{k(n)}(\cdot)'\big]\right)'\widehat{\Sigma}_i^{-1}(\omega_{i,n}-1)^2 \times \rho(Z_i,\widehat{\alpha}_n^B)\rho(Z_i,\widehat{\alpha}_n^B)'\widehat{\Sigma}_i^{-1}\left(\frac{d\widehat{m}^B(X_i,\widehat{\alpha}_n^B)}{d\alpha}\big[\overline{\psi}^{k(n)}(\cdot)'\big]\right).$$

This expression is computed in the same way as $\|\widehat{v}_n^*\|_{n,\mathrm{sd}}^2 = \widehat{F}_n'\widehat{D}_n^{-1}\widehat{\mho}_n\widehat{D}_n^{-1}\widehat{F}_n$ given in (4.9) but using bootstrap analogs. Note that this bootstrap sieve variance only uses $\widehat{\alpha}_n^B$, and is easy to compute.

When specialized to the NPIV model (2.18) in Section 2.2.1, the expression $\|\widehat{v}_n^B\|_{B,\mathrm{sd}}^2$ simplifies further, with $\widehat{F}_n^B = \frac{d\phi(\widehat{h}_n^B)}{d\alpha}[q^{k(n)}(\cdot)']$, $\widehat{D}_n^B = \frac{1}{n}\widehat{C}_n^B(P'P)^-(\widehat{C}_n^B)'$, $\widehat{C}_n^B = \sum_{j=1}^n \omega_{j,n}q^{k(n)}(Y_{2j})p^{J_n}(X_j)'$,

$$\widehat{\mho}_{3,n}^B = \frac{1}{n}\widehat{C}_n^B(P'P)^-\left(\sum_{i=1}^n p^{J_n}(X_i)\big[(\omega_{i,n}-1)\widehat{U}_i^B\big]^2 p^{J_n}(X_i)'\right) \times (P'P)^-(\widehat{C}_n^B)', \quad \text{with} \quad \widehat{U}_i^B = Y_{1i} - \widehat{h}_n^B(Y_{2i}).$$

This expression is analogous to that for a 2SLS t-bootstrap test; see Davidson and MacKinnon (2010). We leave it to further work to study whether this bootstrap sieve t statistic might have second order refinement by choice of some i.i.d. bootstrap weights.

Recall that $\widehat{M}_i^B = (\omega_{i,n}-1)^2\widehat{M}_i$ and $\widehat{M}_i = \widehat{\Sigma}_i^{-1}\rho(Z_i,\widehat{\alpha}_n)\rho(Z_i,\widehat{\alpha}_n)'\widehat{\Sigma}_i^{-1}$.

ASSUMPTION B.3: (i) $\sup_{v_1,v_2\in\overline{\mathbf{V}}_{k(n)}^1} |\langle v_1,v_2\rangle_{B,\Sigma^{-1}} - \langle v_1,v_2\rangle_{n,\Sigma^{-1}}| = o_{P_{V^\infty|Z^\infty}}(1)$ wpa1 (P_{Z^∞});

(ii) $\sup_{v\in\overline{\mathbf{V}}_{k(n)}^1} |\langle v,v\rangle_{B,\widehat{M}^B} - \sigma_\omega^2\langle v,v\rangle_{n,\widehat{M}}| = o_{P_{V^\infty|Z^\infty}}(1)$ wpa1 (P_{Z^∞});

(iii) $\sup_{v\in\overline{\mathbf{V}}_{k(n)}^1} n^{-1}\sum_{i=1}^n(\omega_{i,n}-1)^2\|\frac{d\widehat{m}^B(X_i,\widehat{\alpha}_n^B)}{d\alpha}[v]\|_e^2 = O_{P_{V^\infty|Z^\infty}}(1)$ wpa1 (P_{Z^∞}).

Assumption B.3(i)(ii) is analogous to Assumption 4.1(ii)(v). Assumption B.3(iii) is a mild one; for example, it is implied by assumptions for Lemma A.1 and uniformly bounded bootstrap weights (i.e., $|\omega_{i,n}| \le C < \infty$ for all i).

The following result is a bootstrap version of Theorem 4.2.

THEOREM B.3: *Let conditions for Theorem* 4.2(1) *and Lemma* A.1, *Assumption* B.3 *hold*. *Then*:

(1)

$$\left|\frac{\|\widehat{v}_n^B\|_{B,\mathrm{sd}}}{\sigma_\omega\|v_n^*\|_{\mathrm{sd}}}-1\right|=o_{P_{V^\infty|Z^\infty}}(1)\quad wpa1\ (P_{Z^\infty}).$$

(2) *If*, *further*, *conditions for Theorem* 5.2(1) *hold*, *then*:

$$\widehat{W}_{3,n}^B=-\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}}{\sigma_\omega}+o_{P_{V^\infty|Z^\infty}}(1)\quad wpa1\ (P_{Z^\infty}),$$
$$\left|\mathcal{L}_{V^\infty|Z^\infty}\left(\widehat{W}_{3,n}^B|Z^n\right)-\mathcal{L}(\widehat{W}_n)\right|=o_{P_{Z^\infty}}(1),\quad and$$
$$\sup_{t\in\mathbb{R}}\left|P_{V^\infty|Z^\infty}\left(\widehat{W}_{3,n}^B\le t|Z^n\right)-P_{Z^\infty}(\widehat{W}_n\le t)\right|=o_{P_{V^\infty|Z^\infty}}(1)\quad wpa1\ (P_{Z^\infty}).$$

PROOF: For *Result (1)*, the proof is analogous to the one for Theorem 4.2(1). As in the proof of Theorem 4.2(1), it suffices to show that

$$\frac{\|\widehat{v}_n^B-v_n^*\|}{\|v_n^*\|}=o_{P_{V^\infty|Z^\infty}}(1)\quad \text{wpa1 } (P_{Z^\infty}),\tag{B.35}$$

and

$$\left|\frac{\|\widehat{v}_n^B\|_{B,\mathrm{sd}}-\|\widehat{v}_n^B\|_{\mathrm{sd}}}{\|v_n^*\|}\right|=o_{P_{V^\infty|Z^\infty}}(1)\quad \text{wpa1 } (P_{Z^\infty}).\tag{B.36}$$

Following the same derivations as in the proof of Theorem 4.2(1) Step 1, for equation (B.35), it suffices to show

$$\left|\langle\widehat{\varpi}_n^B,\varpi\rangle_{B,\widehat{\Sigma}^{-1}}-\langle\widehat{\varpi}_n^B,\varpi\rangle_{B,\Sigma^{-1}}\right|=o_{P_{V^\infty|Z^\infty}}(1)\quad\text{and}$$
$$\left|\langle\widehat{\varpi}_n^B,\varpi\rangle_{B,\Sigma^{-1}}-\langle\widehat{\varpi}_n^B,\varpi\rangle_{\Sigma^{-1}}\right|=o_{P_{V^\infty|Z^\infty}}(1)$$

wpa1 (P_{Z^∞}), uniformly over $\varpi\in\overline{\mathbf{V}}_{k(n)}^1$, where $\widehat{\varpi}_n^B=\frac{\widehat{v}_n^B}{\|\widehat{v}_n^B\|}$. The first term follows by Assumptions 4.1(iii) and 3.1(iv) and the fact that $\langle\varpi,\varpi\rangle_{B,\Sigma^{-1}}=O_{P_{V^\infty|Z^\infty}}(1)$ wpa1 (P_{Z^∞}) (by Assumptions B.3(i) and 4.1(ii)). The second term follows directly from these two assumptions.

Regarding equation (B.36), following the same derivations as in the proof of Theorem 4.2 Step 2, it suffices to show that $|\|\widehat{\varpi}_n^B\|_{B,\mathrm{sd}}^2-\|\widehat{\varpi}_n^B\|_{\mathrm{sd}}^2|=o_{P_{V^\infty|Z^\infty}}(1)$

wpa1 (P_{Z^∞}). By the triangle inequality,

$$\begin{aligned}
\sup_{v\in\overline{\mathbf{V}}^1_{k(n)}} \left|\langle v,v\rangle_{B,\widehat{W}^B} - \sigma^2_\omega\langle v,v\rangle_{n,\widehat{M}}\right| &\le \sup_{v\in\overline{\mathbf{V}}^1_{k(n)}} \left|\langle v,v\rangle_{B,\widehat{W}^B} - \langle v,v\rangle_{B,\widehat{M}^B}\right| \\
&\quad + \sup_{v\in\overline{\mathbf{V}}^1_{k(n)}} \left|\langle v,v\rangle_{B,\widehat{M}^B} - \sigma^2_\omega\langle v,v\rangle_{n,\widehat{M}}\right| \\
&\equiv A^B_{1n} + A^B_{2n},
\end{aligned}$$

with $\widehat{W}^B_i \equiv \widehat{\Sigma}^{-1}_i \varrho(V_i,\widehat{\alpha}^B_n)\varrho(V_i,\widehat{\alpha}^B_n)'\widehat{\Sigma}^{-1}_i = (\omega_{i,n}-1)^2\widehat{\Sigma}^{-1}_i\rho(Z_i,\widehat{\alpha}^B_n)\rho(Z_i,\widehat{\alpha}^B_n)'\widehat{\Sigma}^{-1}_i$ and $\widehat{M}^B_i = (\omega_{i,n}-1)^2\widehat{M}_i$ and $\widehat{M}_i = \widehat{\Sigma}^{-1}_i\rho(Z_i,\widehat{\alpha}_n)\rho(Z_i,\widehat{\alpha}_n)'\widehat{\Sigma}^{-1}_i$.

It is easy to see that A^B_{1n} is bounded above by

$$\begin{aligned}
&\sup_x \left\| \widehat{\Sigma}^{-1}(x)\left\{\rho\left(z,\widehat{\alpha}^B_n\right)\rho\left(z,\widehat{\alpha}^B_n\right)' - \rho(z,\widehat{\alpha}_n)\rho(z,\widehat{\alpha}_n)'\right\}\widehat{\Sigma}^{-1}(x)\right\|_e \\
&\qquad \times n^{-1}\sum_{i=1}^n(\omega_{i,n}-1)^2\left\|\widehat{T}^B_i[v]\right\|^2_e \\
&\le 2\sup_x \sup_{\alpha\in\mathcal{N}_{osn}} \left\|\widehat{\Sigma}^{-1}(x)\left\{\rho(z,\alpha)\rho(z,\alpha)' - \rho(z,\alpha_0)\rho(z,\alpha_0)'\right\}\widehat{\Sigma}^{-1}(x)\right\|_e \\
&\qquad \times n^{-1}\sum_{i=1}^n(\omega_{i,n}-1)^2\left\|\widehat{T}^B_i[v]\right\|^2_e,
\end{aligned}$$

where $\widehat{T}^B_i[v] \equiv \frac{d\widehat{m}^B(X_i,\widehat{\alpha}^B_n)}{d\alpha}[v]$. The second line follows because $\widehat{\alpha}^B \in \mathcal{N}_{osn}$ wpa1. The first term in the RHS is of order $o_{P_{Z^\infty}}(1)$ by Assumption 4.1(iv). The second term is $O_{P_{V^\infty|Z^\infty}}(1)$ by Assumption B.3(iii).

A^B_{2n} is of order $o_{P_{V^\infty|Z^\infty}}(1)$ wpa1 (P_{Z^∞}) by Assumption B.3(ii).

Result (1) now follows from the same derivations as in the proof of Theorem 4.2(1) Step 2a.

Given Result (1), *Result (2)* follows from exactly the same proof as that of Theorem 5.2(1), and is omitted. *Q.E.D.*

B.4. *Proofs for Section 6 on Examples*

PROOF OF PROPOSITION 6.1: By our assumption over $\text{clsp}\{p_j : j=1,\dots,J\}$, $\frac{dm(x,\alpha_0)}{d\alpha}[u^*_n] \in \text{clsp}\{p_j : j=1,\dots,J_n\}$ provided $k(n)\le J_n$, and thus Assumption A.6(i) trivially holds. Since $\Sigma = 1$, Assumption A.6(ii) is the same as Assumption A.6(i).

We now show that Assumption A.6(iii)(iv) holds under Condition 6.1. First, Condition 6.1(i) implies that $\{(E[h(Y_2) - h_0(Y_2)|\cdot])^2 : h \in \mathcal{H}\}$ is a P-Donsker class and, moreover,

$$E\big[\big(E\big[h(Y_2) - h_0(Y_2)|X\big]\big)^4\big] \le 2c \times \|h - h_0\|^2 \to 0$$

as $\|h - h_0\|_{L^2(f_{Y_2})} \to 0$. So by Lemma 1 in Chen, Linton, and van Keilegom (2003), Assumption A.6(iii) holds. Regarding Assumption A.6(iv), by Theorem 2.14.2 in VdV-W, (up to omitted constants)

$$\begin{aligned} &E\left[\left|\sup_{f\in\mathcal{F}_n} n^{-1/2}\sum_{i=1}^{n}\big\{f(X_i) - E\big[f(X_i)\big]\big\}\right|\right] \\ &\quad \le \int_0^{\|F_n\|_{L^2(f_X)}} \sqrt{1 + \log N_{[]}\big(u, \mathcal{F}_n, \|\cdot\|_{L^2(f_X)}\big)}\, du \end{aligned}$$

where $\mathcal{F}_n \equiv \{f : f = g(\cdot, u_n^*)(m(\cdot, \alpha) - m(\cdot, \alpha_0)), \text{some } \alpha \in \mathcal{N}_{osn}\}$ and

$$F_n(x) \equiv \sup_{\mathcal{F}_n}\big|f(x)\big| = \sup_{\alpha\in\mathcal{N}_{osn}} \big|g\big(x, u_n^*\big)\big\{m(x, \alpha) - m(x, \alpha_0)\big\}\big|.$$

We claim that, under our assumptions,

$$N_{[]}\big(u, \mathcal{F}_n, \|\cdot\|_{L^2(f_X)}\big) \le N_{[]}\big(u, \Lambda_c^\gamma(\mathcal{X}), \|\cdot\|_{L^\infty}\big).$$

To show this claim, it suffices to show that, given a radius $\delta > 0$, if we take $\{[l_j, u_j]\}_{j=1}^{N(\delta)}$ to be brackets of $\Lambda_c^\gamma(\mathcal{X})$ under $\|\cdot\|_{L^\infty}$, then we can construct $\{[l_{n,j}, u_{n,j}]\}_{j=1}^{N(\delta)}$ such that: they are valid brackets of $\mathcal{F}_n$, under $\|\cdot\|_{L^2(f_X)}$. To show this, observe that, for any $f_n \in \mathcal{F}_n$, there exists an $\alpha \in \mathcal{N}_{osn}$, such that $f_n = g(\cdot, u_n^*)\{m(\cdot, \alpha) - m(\cdot, \alpha_0)\}$, and under Condition 6.1, it follows that there exists a $j \in \{1, \dots, N(\delta)\}$ such that

$$l_j \le m(\cdot, \alpha) - m(\cdot, \alpha_0) \le u_j, \tag{B.37}$$

hence, there exists a $[l_{n,j}, u_{n,j}]$ such that, for all x,

$$l_{n,j}(x) = \big(1\big\{g\big(x, u_n^*\big) > 0\big\}l_j(x) + 1\big\{g\big(x, u_n^*\big) < 0\big\}u_j(x)\big)g\big(x, u_n^*\big),$$

and

$$u_{n,j}(x) = \big(1\big\{g\big(x, u_n^*\big) > 0\big\}u_j(x) + 1\big\{g\big(x, u_n^*\big) < 0\big\}l_j(x)\big)g\big(x, u_n^*\big),$$

such that $l_{n,j} \le f_n \le u_{n,j}$. Also, observe that

$$\begin{aligned} \|l_{n,j} - u_{n,j}\|_{L^2(f_X)} &= \sqrt{E\big[\big(g\big(X, u_n^*\big)\big)^2\big(u_j(X) - l_j(X)\big)^2\big]} \\ &\le \|u_j - l_j\|_{L^\infty} \le \delta \end{aligned}$$

because $E[(g(X,u_n^*))^2]=\|u_n^*\|^2=1$ and $\|u_j-l_j\|_{L^\infty}\le\delta$ by construction.

Therefore,

$$E\left[\left|\sup_{f\in\mathcal{F}_n} n^{-1/2}\sum_{i=1}^n\{f(X_i)-E[f(X_i)]\}\right|\right] \le \int_0^{\|F_n\|_{L^2(f_X)}}\sqrt{1+\log N_{[]}\big(u,\Lambda_c^\gamma(\mathcal{X}),\|\cdot\|_{L^\infty}\big)}\,du.$$

Since by assumption, $\gamma>0.5$, it is well known that $(1+\log N_{[]}(u,\Lambda_c^\gamma(\mathcal{X}),\|\cdot\|_{L^\infty}))^{1/2}$ is integrable, so in order to show that $E[|\sup_{f\in\mathcal{F}_n} n^{-1/2}\sum_{i=1}^n\{f(X_i)-E[f(X_i)]\}|]=o(1)$, it suffices to show that $\|F_n\|_{L^2(f_X)}=o(1)$. In order to show this,

$$\begin{aligned}
&\|F_n\|_{L^2(f_X)}\\
&\le\sqrt{E\Big[\big(g(X,u_n^*)\big)^2\Big(\sup_{\mathcal{N}_{osn}}\big|m(X,\alpha)-m(X,\alpha_0)\big|\Big)^2\Big]}\\
&=\sqrt{E\Big[\big(g(X,u_n^*)\big)^2\Big(\sup_{\mathcal{N}_{osn}}\big|E[h(Y_2)-h_0(Y_2)|X]\big|\Big)^2\Big]}\\
&=\sqrt{E\Big[\big(g(X,u_n^*)\big)^2\sup_{\mathcal{N}_{osn}}\int\big(h(y_2)-h_0(y_2)\big)^2 f_{Y_2|X}(y_2,X)\,dy_2\Big]}\\
&=\Big(E\Big[\big(g(X,u_n^*)\big)^2\\
&\qquad\times\sup_{\mathcal{N}_{osn}}\int\big(h(y_2)-h_0(y_2)\big)^2\frac{f_{Y_2X}(y_2,X)}{f_{Y_2}(y_2)f_X(X)}f_{Y_2}(y_2)\,dy_2\Big]\Big)^{1/2}\\
&\le\sup_{x,y_2}\frac{f_{Y_2X}(y_2,x)}{f_{Y_2}(y_2)f_X(x)}\sup_{\mathcal{N}_{osn}}\|h-h_0\|_{L^2(f_{Y_2})}\sqrt{E\big[\big(g(X,u_n^*)\big)^2\big]}\\
&\le\text{const.}\times M_n\delta_{s,n}\to0,
\end{aligned}$$

where the last expression follows from the fact that $E[(g(X,u_n^*))^2]=\|u_n^*\|^2=1$ and Condition 6.1(ii), that states that

$$\sup_{x,y_2}\frac{f_{Y_2X}(y_2,x)}{f_{Y_2}(y_2)f_X(x)}\le\text{const.}<\infty.$$

Hence, $E[|\sup_{f\in\mathcal{F}_n} n^{-1/2}\sum_{i=1}^n\{f(X_i)-E[f(X_i)]\}|]=o(1)$, which implies Assumption A.6(iv). Finally, Assumption A.7 is automatically satisfied with the NPIV model. *Q.E.D.*

PROOF OF PROPOSITION 6.2: Assumptions A.6(i) and (ii) hold by the same calculations as those in the proof of Proposition 6.1 (for the NPIV model). Also, under Condition 6.2(i), $\{E[F_{Y_1|Y_2X}(h(Y_2), Y_2, \cdot)|\cdot] : h \in \mathcal{H}\} \subseteq \Lambda_c^\gamma(\mathcal{X})$ with $\gamma > 0.5$, Assumptions A.6(iii) and (iv) hold by similar calculations to those in the proof of Proposition 6.1.

Assumption A.7(i) is standard in the literature. Regarding Assumption A.7(ii), observe that for any $h \in \mathcal{N}_{osn}$,

$$\begin{aligned}
&\left|\frac{dm(x,h)}{dh}[u_n^*] - \frac{dm(x,h_0)}{dh}[u_n^*]\right| \\
&\quad = \bigl|E\bigl[\bigl\{f_{Y_1|Y_2X}\bigl(h(Y_2), Y_2, x\bigr) \\
&\qquad - f_{Y_1|Y_2X}\bigl(h_0(Y_2), Y_2, x\bigr)\bigr\}u_n^*(Y_2)|X = x\bigr]\bigr| \\
&\quad = \left|\int\left\{\int_0^1 \frac{df_{Y_1|Y_2X}\bigl(h_0(t)(y_2), y_2, x\bigr)}{dy_1}\right.\right. \\
&\qquad \left.\left. \times \bigl(h(y_2) - h_0(y_2)\bigr)u_n^*(y_2)\,dt\right\} f_{Y_2|X}(y_2, x)\,dy_2\right| \\
&\quad = \left|\int\left(\int_0^1 \frac{df_{Y_1|Y_2X}\bigl(h_0(t)(y_2), y_2, x\bigr)}{dy_1}\,dt\right)\right. \\
&\qquad \left. \times \bigl(h(y_2) - h_0(y_2)\bigr)u_n^*(y_2)f_{Y_2}(y_2)\left(\frac{f_{Y_2X}(y_2, x)}{f_{Y_2}(y_2)f_X(x)}\right)dy_2\right| \\
&\quad = \left|\int \Gamma_1(y_2, x)\Gamma_2(y_2, x)\bigl(h(y_2) - h_0(y_2)\bigr)u_n^*(y_2)f_{Y_2}(y_2)\,dy_2\right| \\
&\quad \le \bigl\|\Gamma_1(\cdot, x)\Gamma_2(\cdot, x)\bigr\|_{L^\infty} \times \|h - h_0\|_{L^2(f_{Y_2})}\bigl\|u_n^*\bigr\|_{L^2(f_{Y_2})},
\end{aligned}$$

where $h_0(t) \equiv h_0 + t\{h - h_0\}$ and $\Gamma_1(y_2, x) \equiv (\int_0^1 \frac{df_{Y_1|Y_2X}(h_0(t)(y_2), y_2, x)}{dy_1}\,dt)$ and $\Gamma_2(y_2, x) \equiv \frac{f_{Y_2X}(y_2, x)}{f_{Y_2}(y_2)f_X(x)}$; the last line follows from the Cauchy–Schwarz inequality.

Under Condition 6.2(ii), it follows that

$$\sup_{y_1, y_2, x}\left|\frac{df_{Y_1|Y_2X}(y_1, y_2, x)}{dy_1}\right| \le C < \infty$$

and, under Condition 6.1(ii), it follows that

$$\sup_{x, y_2}\left|\frac{f_{Y_2X}(y_2, x)}{f_{Y_2}(y_2)f_X(x)}\right| \le C < \infty.$$

Then it is easy to see that $\|\Gamma_j(\cdot, x)\|_{L^\infty(f_{Y_2})} \leq C < \infty$ for both $j = 1, 2$. Thus

$$\left|\frac{dm(x,h)}{dh}[u_n^*] - \frac{dm(x,h_0)}{dh}[u_n^*]\right| \leq C^2 \times \|h - h_0\|_{L^2(f_{Y_2})}\|u_n^*\|_{L^2(f_{Y_2})}$$

and thus, Assumption A.7(ii) is satisfied provided that $n \times M_n^2\delta_n^2 \sup_{h\in\mathcal{N}_{osn}} \|h - h_0\|^2_{L^2(f_{Y_2})}\|u_n^*\|^2_{L^2(f_{Y_2})} = o(1)$. Since $\|u_n^*\|_{L^2(f_{Y_2})} \leq c\mu^{-1}_{k(n)}$, it suffices to show that

$$nM_n^4\delta_n^2\big(\|\Pi_n h_0 - h_0\|_{L^2(f_{Y_2})} + \mu^{-1}_{k(n)}\delta_n\big)^2\mu^{-2}_{k(n)} = o(1).$$

By assumption, $\|\Pi_n h_0 - h_0\|_{L^2(f_{Y_2})} \leq \text{const.} \times \mu^{-1}_{k(n)}\delta_n = O(\delta_{s,n})$ and $\delta_n^2 \asymp \text{const.}\, k(n)/n$, then it suffices to show that

$$nM_n^4\delta_{s,n}^4 = o(1),$$

which holds by Condition 6.3.

Regarding Assumption A.7(iii), observe that for any $h \in \mathcal{N}_{osn}$,

$$\begin{aligned}&\frac{d^2m(x,h)}{dh^2}[u_n^*, u_n^*]\\ &\quad= \int \frac{df_{Y_1|Y_2X}\big(h(y_2), y_2, x\big)}{dy_1}\big(u_n^*(y_2)\big)^2 f_{Y_2|X}(y_2, x)\,dy_2.\end{aligned}$$

Again by Conditions 6.2(ii) and 6.1(ii), it follows that $|\frac{d^2m(x,h)}{dh^2}[u_n^*, u_n^*]| \leq C^2 \times \|u_n^*\|^2_{L^2(f_{Y_2})}$. Since $\|u_n^*\|_{L^2(f_{Y_2})} \leq \text{const.} \times \mu^{-1}_{k(n)}$, Assumption A.7(iii) holds because

$$\mu^{-2}_{k(n)} \times (M_n\delta_n)^2 = o(1), \quad \text{or} \quad M_n^2\delta_{s,n}^2 = o(1).$$

Finally, we verify Assumption A.7(iv). By our previous calculations,

$$\begin{aligned}&\left|\frac{dm(x,h_1)}{dh}[h_2 - h_0] - \frac{dm(x,h_0)}{dh}[h_2 - h_0]\right|\\ &\quad= \bigg|\int\bigg(\int \frac{df_{Y_1|Y_2X}\big(h_0(y_2) + t\big[h_1(y_2) - h_0(y_2)\big], y_2, x\big)}{dy_1}\,dt\bigg)\\ &\qquad\times \big(h_1(y_2) - h_0(y_2)\big)\big(h_2(y_2) - h_0(y_2)\big) f_{Y_2|X}(y_2, x)\,dy_2\bigg|\\ &\quad\leq C^2 \times \int \big|\big(h_1(y_2) - h_0(y_2)\big)\big(h_2(y_2) - h_0(y_2)\big)\big| f_{Y_2}(y_2)\,dy_2\\ &\quad\leq C^2 \times \|h_1 - h_0\|_{L^2(f_{Y_2})}\|h_2 - h_0\|_{L^2(f_{Y_2})},\end{aligned}$$

where the first inequality follows from Conditions 6.2(ii) and 6.1(ii), and the last one from the Cauchy–Schwarz inequality. This result and the Cauchy–Schwarz inequality together imply that

$$\begin{aligned}
&\left|E\left[g\left(X,u_n^*\right)\left(\frac{dm(X,h_1)}{dh}[h_2-h_0]-\frac{dm(X,h_0)}{dh}[h_2-h_0]\right)\right]\right| \\
&\quad\le C^2\sqrt{E\left[\left(g\left(X,u_n^*\right)\right)^2\right]}\|h_1-h_0\|_{L^2(f_{Y_2})}\|h_2-h_0\|_{L^2(f_{Y_2})} \\
&\quad\le \text{const.}\times\|h_1-h_0\|_{L^2(f_{Y_2})}\|h_2-h_0\|_{L^2(f_{Y_2})},
\end{aligned}$$

where the last line follows from $E[(g(X,u_n^*))^2]=\|u^*\|^2\asymp 1$. Thus, Assumption A.7(iv) follows if

$$\delta_{s,n}^2=\left(\|\Pi_n h_0-h_0\|_{L^2(f_{Y_2})}+\mu_{k(n)}^{-1}\delta_n\right)^2=o\left(n^{-1/2}\right)$$

which holds by Condition 6.3. *Q.E.D.*

APPENDIX C: PROOFS OF THE RESULTS IN APPENDIX A

In Appendix C, we provide the proofs of all the lemmas, theorems, and propositions stated in Appendix A.

C.1. *Proofs for Section* A.2 *on Convergence Rates of Bootstrap PSMD Estimators*

PROOF OF LEMMA A.1: For *Result (1)*, we prove this result in two steps. First, we show that $\widehat{\alpha}_n^B\in\mathcal{A}_{k(n)}^{M_0}$ wpa1-$P_{V^\infty|Z^\infty}$ for any Z^∞ in a set that occurs with P_{Z^∞} probability approaching 1, where $\mathcal{A}_{k(n)}^{M_0}$ is defined in the text. Second, we establish consistency, using the fact that we are in the $\mathcal{A}_{k(n)}^{M_0}$ set.

STEP 1: We show that for any $\delta>0$, there exists a $N(\delta)$ such that

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\widehat{\alpha}_n^B\notin\mathcal{A}_{k(n)}^{M_0}|Z^n\right)<\delta\right)\ge 1-\delta,\quad\forall n\ge N(\delta).$$

To show this, note that, by definition of $\widehat{\alpha}_n^B$,

$$\lambda_n\,\mathrm{Pen}\left(\widehat{h}_n^B\right)\le\widehat{Q}_n^B(\widehat{\alpha}_n)+\lambda_n\,\mathrm{Pen}(\widehat{h}_n)+o_{P_{V^\infty|Z^\infty}}\left(\frac{1}{n}\right),\quad\text{wpa1 }(P_{Z^\infty}).$$

By Assumption A.1(i) and the definition of $\widehat{\alpha}_n \in \mathcal{A}_{k(n)}$,

$$\begin{aligned}
&\lambda_n \operatorname{Pen}(\widehat{h}_n^B) \\
&\quad \le c_0^*\big(\widehat{Q}_n(\widehat{\alpha}_n) + \lambda_n \operatorname{Pen}(\widehat{h}_n)\big) + o_{P_{V^\infty|Z^\infty}}\left(\frac{1}{n}\right), \quad \text{wpa1 } (P_{Z^\infty}) \\
&\quad \le c_0^*\big(\widehat{Q}_n(\Pi_n\alpha_0) + \lambda_n \operatorname{Pen}(\Pi_n h_0)\big) + o_{P_{V^\infty|Z^\infty}}\left(\frac{1}{n}\right), \quad \text{wpa1 } (P_{Z^\infty}).
\end{aligned}$$

By Assumptions 3.2(i)(ii) and 3.3(i),

$$\begin{aligned}
\lambda_n \operatorname{Pen}(\widehat{h}_n^B) &\le c_0^* c_0 Q(\Pi_n\alpha_0) + \lambda_n \operatorname{Pen}(h_0) \\
&\quad + O_{P_{V^\infty|Z^\infty}}\left(\lambda_n + o\left(\frac{1}{n}\right)\right), \quad \text{wpa1 } (P_{Z^\infty}).
\end{aligned}$$

By the fact that $Q(\Pi_n\alpha_0) + o(\frac{1}{n}) = O(\lambda_n)$, the desired result follows.

STEP 2: We want to show that for any $\delta > 0$, there exists a $N(\delta)$ such that

$$P_{Z^\infty}\big(P_{V^\infty|Z^\infty}\big(\|\widehat{\alpha}_n^B - \alpha_0\|_s \ge \delta | Z^n\big) < \delta\big) \ge 1 - \delta, \quad \forall n \ge N(\delta),$$

which is equivalent to showing that $P_{Z^\infty}(P_{V^\infty|Z^\infty}(\|\widehat{\alpha}_n^B - \alpha_0\|_s \ge \delta | Z^n) > \delta) \le \delta$ eventually. Note that

$$\begin{aligned}
&P_{Z^\infty}\big(P_{V^\infty|Z^\infty}\big(\|\widehat{\alpha}_n^B - \alpha_0\|_s \ge \delta | Z^n\big) > \delta\big) \\
&\quad \le P_{Z^\infty}\big(P_{V^\infty|Z^\infty}\big(\{\|\widehat{\alpha}_n^B - \alpha_0\|_s \ge \delta\} \cap \{\widehat{\alpha}_n^B \in \mathcal{A}_{k(n)}^{M_0}\} | Z^n\big) > 0.5\delta\big) \\
&\qquad + P_{Z^\infty}\big(P_{V^\infty|Z^\infty}\big(\widehat{\alpha}_n^B \notin \mathcal{A}_{k(n)}^{M_0} | Z^n\big) > 0.5\delta\big).
\end{aligned}$$

By Step 1, the second summand in the RHS is negligible. Thus, it suffices to show that

$$\begin{aligned}
&P_{Z^\infty}\big(P_{V^\infty|Z^\infty}\big(\widehat{\alpha}_n^B \in \mathcal{A}_{k(n)}^{M_0} : \|\widehat{\alpha}_n^B - \alpha_0\|_s \ge \delta | Z^n\big) < \delta\big) \\
&\quad \ge 1 - \delta, \quad \forall n \ge N(\delta)
\end{aligned}$$

(henceforth, we omit $\widehat{\alpha}_n^B \in \mathcal{A}_{k(n)}^{M_0}$). Note that, conditioning on Z^n, by Assumption A.1(i)(ii), the definition of $\widehat{\alpha}_n \in \mathcal{A}_{k(n)}^{M_0}$, Assumption 3.2(i)(ii), and $\max\{\lambda_n, o(\frac{1}{n})\} = O(\lambda_n)$, we have

$$\begin{aligned}
&P_{V^\infty|Z^\infty}\big(\|\widehat{\alpha}_n^B - \alpha_0\|_s \ge \delta | Z^n\big) \\
&\quad \le P_{V^\infty|Z^\infty}\Big(\inf_{\{\mathcal{A}_{k(n)}^{M_0} : \|\alpha - \alpha_0\|_s \ge \delta\}} \big\{\widehat{Q}_n^B(\alpha) + \lambda_n \operatorname{Pen}(h)\big\}
\end{aligned}$$

$$\le \widehat{Q}_n^B(\widehat{\alpha}_n) + \lambda_n \operatorname{Pen}(\widehat{h}_n) + o\left(\frac{1}{n}\right)\Big|Z^n\Big)$$

$$\le P_{V^\infty|Z^\infty}\Big(\inf_{\{\mathcal{A}_{k(n)}^{M_0}:\|\alpha-\alpha_0\|_s\ge\delta\}}\big\{c^*\widehat{Q}_n(\alpha) + \lambda_n \operatorname{Pen}(h)\big\}$$

$$\le c_0^*\big[\widehat{Q}_n(\widehat{\alpha}_n) + \lambda_n \operatorname{Pen}(\widehat{h}_n)\big] + O(\lambda_n) + \overline{\delta}_{m,n}^{*2}\big|Z^n\Big)$$

$$\le P_{V^\infty|Z^\infty}\Big(\inf_{\{\mathcal{A}_{k(n)}^{M_0}:\|\alpha-\alpha_0\|_s\ge\delta\}}\big\{c^*\widehat{Q}_n(\alpha)\big\}$$

$$\le c_0^*\big[\widehat{Q}_n(\Pi_n\alpha_0) + \lambda_n \operatorname{Pen}(\Pi_n h_0)\big] + O(\lambda_n) + \overline{\delta}_{m,n}^{*2}\big|Z^n\Big).$$

Thus, wpa1 (P_{Z^∞}),

$$P_{V^\infty|Z^\infty}\big(\big\|\widehat{\alpha}_n^B - \alpha_0\big\|_s \ge \delta|Z^n\big)$$

$$\le P_{V^\infty|Z^\infty}\Big(\inf_{\{\mathcal{A}_{k(n)}^{M_0}:\|\alpha-\alpha_0\|_s\ge\delta\}} c^*\widehat{Q}_n(\alpha)$$

$$\le c_0^*\widehat{Q}_n(\Pi_n\alpha_0) + M\big(\lambda_n + \overline{\delta}_{m,n}^{*2}\big)\Big|Z^n\Big),$$

which can be bounded above by

$$P_{V^\infty|Z^\infty}\Big(\inf_{\{\mathcal{A}_{k(n)}^{M_0}:\|\alpha-\alpha_0\|_s\ge\delta\}} c^* c Q(\alpha)$$

$$\le c_0^* c_0 Q(\Pi_n\alpha_0) + M\big(\lambda_n + \big(\overline{\delta}_{m,n} + \overline{\delta}_{m,n}^{*}\big)^2\big)\big|Z^n\Big)$$

$$+ P_{V^\infty|Z^\infty}\Big(\sup_{\{\mathcal{A}_{k(n)}^{M_0}:\|\alpha-\alpha_0\|_s\ge\delta\}} \widehat{Q}_n(\alpha) - cQ(\alpha) < -M\overline{\delta}_{m,n}^{2}\big|Z^n\Big)$$

$$+ P_{V^\infty|Z^\infty}\left(\widehat{Q}_n(\Pi_n\alpha_0) - c_0 Q(\Pi_n\alpha_0) > -o\left(\frac{1}{n}\right)\Big|Z^n\right).$$

Therefore, for sufficiently large n,

$$P_{Z^\infty}\big(P_{V^\infty|Z^\infty}\big(\big\|\widehat{\alpha}_n^B - \alpha_0\big\|_s \ge \delta|Z^n\big) < \delta\big)$$

$$\le 0.25\delta + P_{Z^\infty}\Big(\inf_{\{\mathcal{A}_{k(n)}^{M_0}:\|\alpha-\alpha_0\|_s\ge\delta\}} c^* c Q(\alpha)$$

$$\le c_0^* c_0 Q(\Pi_n\alpha_0) + M\big(\lambda_n + \big(\overline{\delta}_{m,n} + \overline{\delta}_{m,n}^{*}\big)^2\big)\Big)$$

$$+ P_{Z^\infty}\Big(\sup_{\{\mathcal{A}^{M_0}_{k(n)}\,:\,\|\alpha-\alpha_0\|_s\geq\delta\}} \widehat{Q}_n(\alpha) - cQ(\alpha) < -M\overline{\delta}^2_{m,n}\Big)$$

$$+ P_{Z^\infty}\left(\widehat{Q}_n(\Pi_n\alpha_0) - c_0Q(\Pi_n\alpha_0) > -o\left(\frac{1}{n}\right)\right).$$

By Assumption 3.3, the third and fourth terms in the RHS are less than 0.5δ. The second term in the RHS is not random. By Assumptions 3.1(ii) and 3.2(iii), $\mathcal{A}^{M_0}_{k(n)}$ is compact, and so is $\mathcal{A}^{M_0} \equiv \{\alpha = (\theta', h) \in \mathcal{A} : \lambda_n \operatorname{Pen}(h) \leq \lambda_n M_0\}$. This fact and Assumption 3.1(iii) imply that $\inf_{\{\mathcal{A}^{M_0}_{k(n)}\,:\,\|\alpha-\alpha_0\|_s\geq\delta\}} Q(\alpha) \geq Q(\alpha(\delta))$ for some $\alpha(\delta) \in \mathcal{A}^{M_0} \cap \{\|\alpha - \alpha_0\|_s \geq \delta\}$. By Assumption 3.1(i), $Q(\alpha(\delta)) > 0$, so eventually, since $c_0^* c_0 Q(\Pi_n\alpha_0) + M(\lambda_n + (\overline{\delta}_{m,n} + \overline{\delta}^*_{m,n})^2) = o(1)$,

$$P_{Z^\infty}\Big(\inf_{\{\mathcal{A}^{M_0}_{k(n)}\,:\,\|\alpha-\alpha_0\|_s\geq\delta\}} c^*cQ(\alpha)$$
$$\leq c_0^* c_0 Q(\Pi_n\alpha_0) + M\big(\lambda_n + \big(\overline{\delta}_{m,n} + \overline{\delta}^*_{m,n}\big)^2\big)\Big) = 0.$$

For *Result (2)*, we want to show that for any $\delta > 0$, there exists a $M(\delta)$ such that

$$P_{Z^\infty}\big(P_{V^\infty|Z^\infty}\big(\delta_n^{-1}\big\|\widehat{\alpha}^B_n - \alpha_0\big\| \geq M'|Z^n\big) < \delta\big) \geq 1 - \delta, \quad \forall M' \geq M(\delta)$$

eventually. By Assumptions 3.4(iii) and A.1(iii), following the similar algebra as before, we have: for M' large enough,

$$P_{V^\infty|Z^\infty}\big(\delta_n^{-1}\big\|\widehat{\alpha}^B_n - \alpha_0\big\| \geq M'|Z^n\big)$$
$$\leq P_{V^\infty|Z^\infty}\Big(\inf_{\{\mathcal{A}_{osn}:\delta_n^{-1}\|\alpha-\alpha_0\|\geq M'\}} c^*cQ(\alpha) \leq M\big(\lambda_n + \delta_n^2\big)\big|Z^n\Big) + \delta.$$

By Assumption 3.4(i)(ii) and $\delta_n = \sqrt{\max\{\lambda_n, \delta_n^2\}}$, we have

$$P_{V^\infty|Z^\infty}\Big(\inf_{\{\mathcal{A}_{osn}:\delta_n^{-1}\|\alpha-\alpha_0\|\geq M'\}} c^*cQ(\alpha) \leq M\big(\lambda_n + \delta_n^2\big)\big|Z^n\Big)$$
$$\leq 1\big\{c^*cc_1\big(M'\delta_n\big)^2 \leq M\big(\lambda_n + \delta_n^2\big)\big\},$$

which is eventually naught, because M' can be chosen to be large. The rate under $\|\cdot\|_s$ immediately follows from this result and the definition of the sieve measure of local ill-posedness τ_n.

For *Result (3)*, we note that both $\widehat{\alpha}_n^{R,B}, \widehat{\alpha}_n \in \{\alpha \in \mathcal{A}_{k(n)} : \phi(\alpha) = \phi(\widehat{\alpha}_n)\}$, and hence all the above proofs go through with $\widehat{\alpha}_n^{R,B}$ replacing $\widehat{\alpha}_n^B$. In particular, let $\mathcal{A}_{k(n)}^{M_0}(\widehat{\phi}) \equiv \{\alpha \in \mathcal{A}_{k(n)}^{M_0} : \phi(\alpha) = \phi(\widehat{\alpha}_n)\} \subseteq \mathcal{A}_{k(n)}^{M_0}$. Then: for any $\delta > 0$,

$$
\begin{aligned}
& P_{V^\infty|Z^\infty}\big(\widehat{\alpha}_n^{R,B} \in \mathcal{A}_{k(n)}^{M_0}(\widehat{\phi}) : \big\|\widehat{\alpha}_n^{R,B} - \alpha_0\big\|_s \geq \delta | Z^n\big) \\
&\quad \leq P_{V^\infty|Z^\infty}\Bigg(\inf_{\{\mathcal{A}_{k(n)}^{M_0}(\widehat{\phi}) : \|\alpha-\alpha_0\|_s \geq \delta\}} \big\{\widehat{Q}_n^B(\alpha) + \lambda_n \operatorname{Pen}(h)\big\} \\
&\qquad \leq \widehat{Q}_n^B(\widehat{\alpha}_n) + \lambda_n \operatorname{Pen}(\widehat{h}_n) + o\bigg(\frac{1}{n}\bigg)\bigg| Z^n\Bigg) \\
&\quad \leq P_{V^\infty|Z^\infty}\Big(\inf_{\{\mathcal{A}_{k(n)}^{M_0}(\widehat{\phi}) : \|\alpha-\alpha_0\|_s \geq \delta\}} \big\{c^* \widehat{Q}_n(\alpha) + \lambda_n \operatorname{Pen}(h)\big\} \leq A_n \big| Z^n\Big) \\
&\quad \leq P_{V^\infty|Z^\infty}\Big(\inf_{\{\mathcal{A}_{k(n)}^{M_0} : \|\alpha-\alpha_0\|_s \geq \delta\}} \big\{c^* \widehat{Q}_n(\alpha)\big\} \\
&\qquad \leq c_0^*\big[\widehat{Q}_n(\Pi_n \alpha_0) + \lambda_n \operatorname{Pen}(\Pi_n h_0)\big] + O(\lambda_n) + \overline{\delta}_{m,n}^{*2} \big| Z^n\Big),
\end{aligned}
$$

where $A_n \equiv c_0^*[\widehat{Q}_n(\widehat{\alpha}_n) + \lambda_n \operatorname{Pen}(\widehat{h}_n)] + O(\lambda_n) + \overline{\delta}_{m,n}^{*2}$. The rest follows from the proof of Results (1) and (2). *Q.E.D.*

C.2. *Proofs for Section A.3 on Behaviors Under Local Alternatives*

PROOF OF THEOREM A.1: The proof is analogous to that of Theorem 4.3, hence we only present the main steps. Let $\boldsymbol{\alpha}_n = \alpha_0 + d_n \Delta_n$ with $\frac{d\phi(\alpha_0)}{d\alpha}[\Delta_n] = \langle v_n^*, \Delta_n \rangle = \kappa_n = \kappa \times (1 + o(1)) \neq 0$.

STEP 1: By Assumption 3.6(i) under the local alternatives, for any $t_n \in \mathcal{T}_n$,

$$
\begin{aligned}
\text{(C.1)} \qquad & 0 \leq 0.5\big(\widehat{Q}_n\big(\widehat{\alpha}_n(t_n)\big) - \widehat{Q}_n(\widehat{\alpha}_n)\big) \\
&= t_n\big\{\mathbb{Z}_n(\boldsymbol{\alpha}_n) + \big\langle u_n^*, \widehat{\alpha}_n - \boldsymbol{\alpha}_n\big\rangle\big\} + \frac{B_n}{2} t_n^2 + o_{P_{n,Z^\infty}}\big(\big[r_n(t_n)\big]^{-1}\big),
\end{aligned}
$$

where $[r_n(t_n)]^{-1} = \max\{t_n^2, t_n n^{-1/2}, s_n^{-1}\}$ and $s_n^{-1} = o(n^{-1})$. The LHS is always positive (up to possibly a negligible term given by the penalty function; see the proof of Theorem 4.1(1) for details) by definition of $\widehat{\alpha}_n$. Hence, by choosing $t_n = \pm\{s_n^{-1/2} + o(n^{-1/2})\}$, it follows that $\{\mathbb{Z}_n(\boldsymbol{\alpha}_n) + \langle u_n^*, \widehat{\alpha}_n - \boldsymbol{\alpha}_n\rangle\} = o_{P_{n,Z^\infty}}(n^{-1/2})$. Since $\langle u_n^*, \boldsymbol{\alpha}_n - \alpha_0\rangle = \frac{d_n \kappa_n}{\|v_n^*\|_{\text{sd}}}$ by the definition of local alternatives $\boldsymbol{\alpha}_n$, we obtain equation (C.2):

$$\text{(C.2)}\qquad \left\{\mathbb{Z}_n(\boldsymbol{\alpha}_n)+\langle u_n^*,\widehat{\alpha}_n-\alpha_0\rangle-\frac{d_n\boldsymbol{\kappa}_n}{\|v_n^*\|_{\mathrm{sd}}}\right\} = \mathbb{Z}_n(\boldsymbol{\alpha}_n)+\langle u_n^*,\widehat{\alpha}_n-\boldsymbol{\alpha}_n\rangle = o_{P_{n,Z^\infty}}(n^{-1/2}),$$

where $\mathbb{Z}_n(\boldsymbol{\alpha}_n)$ is defined as that of $\mathbb{Z}_n$ but using $\rho(z,\boldsymbol{\alpha}_n)$ instead of $\rho(z,\alpha_0)$ (since $m(X,\boldsymbol{\alpha}_n)=0$ a.s.-X under the local alternative).

Next, by Assumption 3.6(i) under the local alternative, we have: for any $t_n\in\mathcal{T}_n$,

$$\text{(C.3)}\qquad 0.5\big(\widehat{Q}_n(\widehat{\alpha}_n^R(t_n))-\widehat{Q}_n(\widehat{\alpha}_n^R)\big) = t_n\{\mathbb{Z}_n(\boldsymbol{\alpha}_n)+\langle u_n^*,\widehat{\alpha}_n^R-\boldsymbol{\alpha}_n\rangle\}+\frac{B_n}{2}t_n^2+o_{P_{n,Z^\infty}}\big([r_n(t_n)]^{-1}\big).$$

By Assumption 3.5(ii),

$$\sup_{\alpha\in\mathcal{N}_{0n}}\left|\phi(\alpha)-\phi(\alpha_0)-\frac{d\phi(\alpha_0)}{d\alpha}[\alpha-\alpha_0]\right| = o\big(n^{-1/2}\|v_n^*\|\big),$$

and assumption $\widehat{\alpha}_n^R\in\mathcal{N}_{osn}$ wpa1-P_{n,Z^∞}, and the fact that $\phi(\widehat{\alpha}_n^R)-\phi(\alpha_0)=0$, following the same calculations as those in Step 1 of the proof of Theorem 4.3, we have

$$\langle u_n^*,\widehat{\alpha}_n^R-\alpha_0\rangle = o_{P_{n,Z^\infty}}(n^{-1/2}).$$

Since $\boldsymbol{\alpha}_n=\alpha_0+d_n\Delta_n\in\mathcal{N}_{osn}$ with $\frac{d\phi(\alpha_0)}{d\alpha}[\Delta_n]=\langle v_n^*,\Delta_n\rangle=\boldsymbol{\kappa}_n$, we have

$$\langle u_n^*,\widehat{\alpha}_n^R-\boldsymbol{\alpha}_n\rangle = \langle u_n^*,\widehat{\alpha}_n^R-\alpha_0\rangle-\frac{d_n\boldsymbol{\kappa}_n}{\|v_n^*\|_{\mathrm{sd}}}+o_{P_{n,Z^\infty}}(n^{-1/2}) = -\frac{d_n\boldsymbol{\kappa}_n}{\|v_n^*\|_{\mathrm{sd}}}+o_{P_{n,Z^\infty}}(n^{-1/2}).$$

Therefore, by choosing $t_n\equiv-(\mathbb{Z}_n(\boldsymbol{\alpha}_n)-\frac{d_n\kappa_n}{\|v_n^*\|_{\mathrm{sd}}})B_n^{-1}$ in (C.3) with $[r_n(t_n)]^{-1}=\max\{t_n^2,t_nn^{-1/2},o(n^{-1})\}$ (which is a valid choice), we obtain

$$\begin{aligned}
&0.5\big(\widehat{Q}_n(\widehat{\alpha}_n)-\widehat{Q}_n(\widehat{\alpha}_n^R)\big)\\
&\quad\le 0.5\big(\widehat{Q}_n(\widehat{\alpha}_n^R(t_n))-\widehat{Q}_n(\widehat{\alpha}_n^R)\big)+o_{P_{n,Z^\infty}}(n^{-1})\\
&\quad=-\frac{1}{2}\left(\frac{\left(\mathbb{Z}_n(\boldsymbol{\alpha}_n)-\frac{d_n\boldsymbol{\kappa}_n}{\|v_n^*\|_{\mathrm{sd}}}\right)}{\sqrt{B_n}}\right)^2+o_{P_{n,Z^\infty}}\big([r_n(t_n)]^{-1}\big).
\end{aligned}$$

By our assumption and the fact that $\|u_n^*\| \geq c > 0$ for all n, it follows that $B_n \geq c > 0$ eventually, so

$$
\begin{aligned}
&0.5\big(\widehat{Q}_n(\widehat{\alpha}_n) - \widehat{Q}_n\big(\widehat{\alpha}_n^R\big)\big) \\
&\quad \leq -\frac{1}{2}\left(\frac{\left(\mathbb{Z}_n(\boldsymbol{\alpha}_n) - \frac{d_n\kappa_n}{\|v_n^*\|_{\mathrm{sd}}}\right)}{\|u_n^*\|}\right)^2 \times \big(1 + o_{P_{n,Z^\infty}}(1)\big).
\end{aligned}
$$

STEP 2: On the other hand, suppose there exists a t_n^*, such that (a) $\phi(\widehat{\alpha}_n(t_n^*)) = \phi(\alpha_0)$, $\widehat{\alpha}_n(t_n^*) \in \mathcal{A}_{k(n)}$, and (b) $t_n^* = (\mathbb{Z}_n(\boldsymbol{\alpha}_n) - \frac{d_n\kappa_n}{\|v_n^*\|_{\mathrm{sd}}})(\|u_n^*\|)^{-2} + o_{P_{n,Z^\infty}}(n^{-1/2})$. Substituting this into (C.1) with $[r_n(t_n^*)]^{-1} = \max\{(t_n^*)^2, t_n^* n^{-1/2}, o(n^{-1})\}$, we obtain

$$
\begin{aligned}
&0.5\big(\widehat{Q}_n(\widehat{\alpha}_n) - \widehat{Q}_n\big(\widehat{\alpha}_n^R\big)\big) \\
&\quad \geq 0.5\big(\widehat{Q}_n(\widehat{\alpha}_n) - \widehat{Q}_n\big(\widehat{\alpha}_n\big(t_n^*\big)\big)\big) - o_{P_{n,Z^\infty}}\big(n^{-1}\big) \\
&\quad = -\frac{B_n}{2}\big(t_n^*\big)^2 + o_{P_{n,Z^\infty}}\big(\big[r_n\big(t_n^*\big)\big]^{-1}\big) \\
&\quad = -\frac{B_n}{2}\left(\mathbb{Z}_n(\boldsymbol{\alpha}_n) - \frac{d_n\kappa_n}{\|v_n^*\|_{\mathrm{sd}}}\right)^2 \big(\|u_n^*\|\big)^{-4} + o_{P_{n,Z^\infty}}\big(\big[r_n\big(t_n^*\big)\big]^{-1}\big) \\
&\quad = -\frac{1}{2}\left(\frac{\mathbb{Z}_n(\boldsymbol{\alpha}_n) - \frac{d_n\kappa_n}{\|v_n^*\|_{\mathrm{sd}}}}{\|u_n^*\|}\right)^2 \times \big(1 + o_{P_{n,Z^\infty}}(1)\big),
\end{aligned}
$$

where the second line follows from equation (C.2). Finally, we observe that point (a) follows from Lemma B.2, with $r = 0$. Point (b) follows by analogous calculations to those in Step 3 of the proof of Theorem 4.3, except that now with $\widehat{\alpha}(t_n^*) = \widehat{\alpha}_n + t_n^* u_n^*$,

$$
\begin{aligned}
&\phi\big(\widehat{\alpha}\big(t_n^*\big)\big) - \phi(\alpha_0) \\
&\quad = \frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n - \alpha_0] + t_n^* \frac{\|v_n^*\|^2}{\|v_n^*\|_{\mathrm{sd}}} + o_{P_{n,Z^\infty}}\big(n^{-1/2}\|v_n^*\|\big) \\
&\quad = -\mathbb{Z}_n(\boldsymbol{\alpha}_n)\|v_n^*\|_{\mathrm{sd}} + \frac{d_n\kappa_n}{\|v_n^*\|_{\mathrm{sd}}}\|v_n^*\|_{\mathrm{sd}} \\
&\qquad + \left(\left(\mathbb{Z}_n(\boldsymbol{\alpha}_n) - \frac{d_n\kappa_n}{\|v_n^*\|_{\mathrm{sd}}}\right)\frac{\|v_n^*\|_{\mathrm{sd}}^2}{\|v_n^*\|^2}\right)\frac{\|v_n^*\|^2}{\|v_n^*\|_{\mathrm{sd}}} + o_{P_{n,Z^\infty}}\big(n^{-1/2}\|v_n^*\|\big) \\
&\quad = o_{P_{n,Z^\infty}}\big(n^{-1/2}\|v_n^*\|\big)
\end{aligned}
$$

where the second line follows from equation (C.2) and some straightforward algebra.

STEP 3: Finally, the above calculations and $\kappa_n = \kappa(1+o(1))$ imply that

$$\|u_n^*\|^2 \times \bigl(\widehat{Q}_n(\widehat{\alpha}_n^R) - \widehat{Q}_n(\widehat{\alpha}_n)\bigr) = \left(\mathbb{Z}_n(\boldsymbol{\alpha}_n) - \frac{d_n \kappa\bigl(1+o(1)\bigr)}{\|v_n^*\|_{\mathrm{sd}}}\right)^2 \times \bigl(1 + o_{P_{n,Z^\infty}}(1)\bigr). \tag{C.4}$$

For *Result (1)*, equation (C.4) with $d_n = n^{-1/2}\|v_n^*\|_{\mathrm{sd}}$ implies that

$$\begin{aligned}\|u_n^*\|^2 \times \widehat{\mathrm{QLR}}_n(\phi_0) &= \bigl(\sqrt{n}\mathbb{Z}_n(\boldsymbol{\alpha}_n) - \kappa\bigl(1+o(1)\bigr)\bigr)^2 \times \bigl(1 + o_{P_{n,Z^\infty}}(1)\bigr) \\ &\Rightarrow \chi_1^2\bigl(\kappa^2\bigr),\end{aligned}$$

which is due to $\sqrt{n}\mathbb{Z}_n(\boldsymbol{\alpha}_n) \Rightarrow N(0,1)$ under the local alternatives.

For *Result (2)*, equation (C.4) with $\sqrt{n}\frac{d_n}{\|v_n^*\|_{\mathrm{sd}}} \to \infty$ implies that

$$\begin{aligned}&\|u_n^*\|^2 \times \widehat{\mathrm{QLR}}_n(\phi_0) \\ &\quad = \left(\sqrt{n}\mathbb{Z}_n(\boldsymbol{\alpha}_n) - \sqrt{n}\frac{d_n \kappa\bigl(1+o(1)\bigr)}{\|v_n^*\|_{\mathrm{sd}}}\right)^2 \times \bigl(1 + o_{P_{n,Z^\infty}}(1)\bigr) \\ &\quad = \left(O_{P_{n,Z^\infty}}(1) - \sqrt{n}\frac{d_n \kappa\bigl(1+o(1)\bigr)}{\|v_n^*\|_{\mathrm{sd}}}\right)^2 \times \bigl(1 + o_{P_{n,Z^\infty}}(1)\bigr),\end{aligned}$$

where the second line is due to $\sqrt{n}\mathbb{Z}_n(\boldsymbol{\alpha}_n) \Rightarrow N(0,1)$ under the local alternatives. Since $\sqrt{n}\frac{d_n\kappa(1+o(1))}{\|v_n^*\|_{\mathrm{sd}}} \to \infty$ (or $-\infty$) if $\kappa > 0$ (or $\kappa < 0$), we have that $\lim_{n\to\infty}(\|u_n^*\|^2 \times \widehat{\mathrm{QLR}}_n(\phi_0)) = \infty$ in probability (under the alternative). *Q.E.D.*

PROOF OF PROPOSITION A.1: Recall that $\widehat{\mathrm{QLR}}{}_n^0(\phi_0)$ denotes the optimally weighted SQLR statistic. By inspection of the proof of Theorem A.1, it is easy to see that

$$\|u_n^*\|^2 \times \widehat{\mathrm{QLR}}_n(\phi_0) = \bigl(\sqrt{n}\mathbb{Z}_n(\boldsymbol{\alpha}_n) - \kappa\bigr)^2 + o_{P_{n,Z^\infty}}(1)$$

and

$$\widehat{\mathrm{QLR}}{}_n^0(\phi_0) = \left(\sqrt{n}\mathbb{Z}_n(\boldsymbol{\alpha}_n) - \kappa\frac{\|v_n^*\|_{\mathrm{sd}}}{\|v_n^0\|_0}\right)^2 + o_{P_{n,Z^\infty}}(1)$$

for local alternatives of the form described in equation (A.2) with $d_n = n^{-1/2}\|v_n^*\|_{\mathrm{sd}}$. Hence, the distribution of $\|u_n^*\|^2 \times \widehat{\mathrm{QLR}}_n(\phi_0)$ is asymptotically close to $\chi_1^2(\kappa^2)$ and the distribution of $\widehat{\mathrm{QLR}}_n^0(\phi_0)$ is asymptotically close to $\chi_1^2(\frac{\|v_n^*\|_{\mathrm{sd}}^2}{\|v_n^0\|_0^2}\kappa^2)$.

Let $A_n^0(z) \equiv (\frac{dm(x,\alpha_0)}{d\alpha}[v_n^0])'(\Sigma_0(x))^{-1}\rho(z,\alpha_0)$ and $A_n(z) \equiv (\frac{dm(x,\alpha_0)}{d\alpha}[v_n^*])' \times (\Sigma(x))^{-1}\rho(z,\alpha_0)$, where v_n^0 is the Riesz representer under $\|\cdot\|_0$. Since $\frac{\langle v_n^*, v_n^0\rangle}{\langle v_n^0, v_n^0\rangle_0}E[(A_n^0(Z))(A_n(Z))'] = \frac{\langle v_n^*, v_n^0\rangle}{\langle v_n^0, v_n^0\rangle_0}E[(\frac{dm(X,\alpha_0)}{d\alpha}[v_n^0])'(\Sigma(X))^{-1}(\frac{dm(X,\alpha_0)}{d\alpha}[v_n^*])] = \frac{(\langle v_n^*, v_n^0\rangle)^2}{\langle v_n^0, v_n^0\rangle_0}$ and $E[(A_n^0(Z))(A_n^0(Z))'] = \langle v_n^0, v_n^0\rangle_0$, we have

$$E\left[\left(A_n(Z) - \frac{\langle v_n^*, v_n^0\rangle}{\langle v_n^0, v_n^0\rangle_0}A_n^0(Z)\right)\left(A_n(Z) - \frac{\langle v_n^*, v_n^0\rangle}{\langle v_n^0, v_n^0\rangle_0}A_n^0(Z)\right)'\right]$$
$$= E\big[\big(A_n(Z)\big)\big(A_n(Z)\big)'\big] - \frac{\big(\langle v_n^*, v_n^0\rangle\big)^2}{\langle v_n^0, v_n^0\rangle_0} = \langle v_n^*, v_n^*\rangle_{\mathrm{sd}} - \frac{\big(\langle v_n^*, v_n^0\rangle\big)^2}{\langle v_n^0, v_n^0\rangle_0}.$$

Since the LHS is nonnegative, the previous equation implies that $\|v_n^*\|_{\mathrm{sd}}^2 - \frac{(\langle v_n^*, v_n^0\rangle)^2}{\langle v_n^0, v_n^0\rangle_0} \geq 0$. By definition of v_n^* and v_n^0, it follows that

$$\langle v_n^*, v_n^0\rangle = \frac{d\phi(\alpha_0)}{d\alpha}\big[v_n^0\big] = \big\|v_n^0\big\|_0^2,$$

and thus $\|v_n^*\|_{\mathrm{sd}}^2 \geq \|v_n^0\|_0^2$ for all n.

Observe that for a noncentral chi-square, $\chi_p^2(r)$, $\Pr(\chi_p^2(r) \leq t)$ is decreasing in the noncentrality parameter r for each t; thus $\Pr(\chi_p^2(r_1) > t) > \Pr(\chi_p^2(r_2) > t)$ for $r_1 > r_2$. Therefore, the previous results imply that, for any t,

$$\begin{aligned}
&\lim_{n\to\infty} P_{n,Z^\infty}\big(\big\|u_n^*\big\|^2 \times \widehat{\mathrm{QLR}}_n(\phi_0) \geq t\big)\\
&\quad = \Pr\big(\chi_1^2\big(\kappa^2\big) \geq t\big)\\
&\quad \leq \liminf_{n\to\infty} \Pr\left(\chi_1^2\left(\frac{\|v_n^*\|_{\mathrm{sd}}^2}{\|v_n^0\|_0^2}\kappa^2\right) \geq t\right)\\
&\quad = \liminf_{n\to\infty} P_{n,Z^\infty}\big(\widehat{\mathrm{QLR}}_n^0(\phi_0) \geq t\big).
\end{aligned}$$

Q.E.D.

PROOF OF THEOREM A.2: The proof of *Result (1)* is similar to that of Theorem 5.3, so we only present a sketch here. By Assumptions 3.6(i) and Boot.3(i) under local alternative, it follows that

$$0.5\left(\widehat{Q}_n^B\left(\widehat{\alpha}_n^{R,B}\left(-\frac{\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)}{B_n^\omega}\right)\right)-\widehat{Q}_n^B\left(\widehat{\alpha}_n^{R,B}\right)\right)$$
$$=-\frac{\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)}{B_n^\omega}\left\{\mathbb{Z}_n^\omega(\boldsymbol{\alpha}_n)+\langle u_n^*,\widehat{\alpha}_n^{R,B}-\boldsymbol{\alpha}_n\rangle\right\}$$
$$+\frac{\left(\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)\right)^2}{2B_n^\omega}+o_{P_{V^\infty|Z^\infty}}\left(r_n^{-1}\right),\quad \text{wpa1 } (P_{n,Z^\infty}),$$

where $r_n^{-1} = \max\{(-\frac{\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)}{B_n^\omega})^2, |-\frac{\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)}{B_n^\omega}|n^{-1/2}, o(n^{-1})\} = O_{P_{V^\infty|Z^\infty}}(n^{-1})$, wpa1 (P_{n,Z^∞}) under Assumption Boot.3(i)(ii) with $\boldsymbol{\alpha}_n$ (instead of α_0).

By similar calculations to those in the proof of Result (1) of Theorem 5.3 (equation (B.23)),

$$\sqrt{n}\langle u_n^*,\widehat{\alpha}_n^{R,B}-\widehat{\alpha}_n\rangle=o_{P_{V^\infty|Z^\infty}}(1),\quad \text{wpa1 } (P_{n,Z^\infty}),$$

that is, the restricted bootstrap estimator $\widehat{\alpha}_n^{R,B}$ centers at $\widehat{\alpha}_n$, regardless of the local alternative. Thus

$$\langle u_n^*,\widehat{\alpha}_n^{R,B}-\boldsymbol{\alpha}_n\rangle=\langle u_n^*,\widehat{\alpha}_n^{R,B}-\widehat{\alpha}_n\rangle+\langle u_n^*,\widehat{\alpha}_n-\boldsymbol{\alpha}_n\rangle$$
$$=\langle u_n^*,\widehat{\alpha}_n-\boldsymbol{\alpha}_n\rangle+o_{P_{V^\infty|Z^\infty}}\left(n^{-1/2}\right),\quad \text{wpa1 } (P_{n,Z^\infty}).$$

This result and equation (C.2) (i.e., $\mathbb{Z}_n(\boldsymbol{\alpha}_n)+\langle u_n^*,\widehat{\alpha}_n-\boldsymbol{\alpha}_n\rangle=o_{P_{n,Z^\infty}}(n^{-1/2})$) imply that

$$0.5\left(\widehat{Q}_n^B\left(\widehat{\alpha}_n^{R,B}\left(-\frac{\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)}{B_n^\omega}\right)\right)-\widehat{Q}_n^B\left(\widehat{\alpha}_n^{R,B}\right)\right)$$
$$=-\frac{\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)}{B_n^\omega}\left\{\mathbb{Z}_n^\omega(\boldsymbol{\alpha}_n)+\langle u_n^*,\widehat{\alpha}_n-\boldsymbol{\alpha}_n\rangle\right\}$$
$$+\frac{\left(\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)\right)^2}{2B_n^\omega}+o_{P_{V^\infty|Z^\infty}}\left(r_n^{-1}\right),\quad \text{wpa1 } (P_{n,Z^\infty})$$
$$=-\frac{\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)}{B_n^\omega}\left\{\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)+o_{P_{n,Z^\infty}}\left(n^{-1/2}\right)\right\}$$
$$+\frac{\left(\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)\right)^2}{2B_n^\omega}+o_{P_{V^\infty|Z^\infty}}\left(r_n^{-1}\right),\quad \text{wpa1 } (P_{n,Z^\infty})$$
$$=-\frac{\left(\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)\right)^2}{2B_n^\omega}\times\left(1+o_{P_{V^\infty|Z^\infty}}(1)\right)\quad \text{wpa1 } (P_{n,Z^\infty}).$$

Following the proof of Result (1) of Theorem 5.3 Step 3 with $\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)$ replacing $\mathbb{Z}_n^{\omega-1}$, we obtain

$$\begin{aligned}\frac{\widehat{\mathrm{QLR}}_n^B(\widehat{\phi}_n)}{\sigma_\omega^2} &= \left(\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)}{\sigma_\omega\sqrt{B_n^\omega}}\right)^2 \times \big(1+o_{P_{V^\infty|Z^\infty}}(1)\big)\\ &= O_{P_{V^\infty|Z^\infty}}(1), \quad \text{wpa1 } (P_{n,Z^\infty}).\end{aligned}$$

This shows that, since for the bootstrap SQLR the "null hypothesis is $\phi(\alpha) = \widehat{\phi}_n \equiv \phi(\widehat{\alpha}_n)$," it always centers correctly.

By similar calculations to those in the proof of Result (2) of Theorem 5.3, the law of $(\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)}{\sigma_\omega\sqrt{B_n^\omega}})^2$ is asymptotically (and wpa1 (P_{n,Z^∞})) equal to the law of $(\frac{\mathbb{Z}}{\|u_n^*\|})^2$ where $\mathbb{Z} \sim N(0,1)$. This implies that the ath quantile of the distribution of $\frac{\widehat{\mathrm{QLR}}_n^B(\widehat{\phi}_n)}{\sigma_\omega^2}$, $\widehat{c}_n(a)$, is uniformly bounded wpa1 (P_{n,Z^∞}). Also, following the proof of Result (2) of Theorem 5.3, we obtain

$$\begin{aligned}&\sup_{t\in\mathbb{R}}\left|P_{V^\infty|Z^\infty}\left(\frac{\widehat{\mathrm{QLR}}_n^B(\widehat{\phi}_n)}{\sigma_\omega^2} \le t\,\Big|\, Z^n\right) - P_{Z^\infty}\big(\widehat{\mathrm{QLR}}_n(\phi_0) \le t | H_0\big)\right|\\ &\quad= o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{n,Z^\infty}).\end{aligned}$$

This and Theorem A.1 (and the fact that $\|u_n^*\| \le c < \infty$) immediately imply *Result (2)*. *Q.E.D.*

PROOF OF THEOREM A.3: The proof is analogous to that of Theorems 4.2 and A.1, so we only present a sketch here.

Under our assumptions, Theorem 4.2 still holds under the local alternatives $\boldsymbol{\alpha}_n$. Observe that, with $\boldsymbol{\alpha}_n = \alpha_0 + d_n\Delta_n \in \mathcal{N}_{osn}$ and $d_n = o(1)$,

$$\begin{aligned}\mathcal{T}_n &\equiv \sqrt{n}\frac{\phi(\widehat{\alpha}_n)-\phi_0}{\|\widehat{v}_n^*\|_{n,\mathrm{sd}}} = \sqrt{n}\frac{\phi(\widehat{\alpha}_n)-\phi_0}{\|v_n^*\|_{\mathrm{sd}}} \times \big(1+o_{P_{n,Z^\infty}}(1)\big)\\ &= \sqrt{n}\langle u_n^*, \widehat{\alpha}_n - \alpha_0\rangle \times \big(1+o_{P_{n,Z^\infty}}(1)\big) + o_{P_{n,Z^\infty}}(1)\\ &= \left(-\sqrt{n}\mathbb{Z}_n(\boldsymbol{\alpha}_n) + \sqrt{n}\frac{d_n\kappa\big(1+o(1)\big)}{\|v_n^*\|_{\mathrm{sd}}}\right) \times \big(1+o_{P_{n,Z^\infty}}(1)\big)\\ &\quad + o_{P_{n,Z^\infty}}(1),\end{aligned}$$

where the second line follows from Assumption 3.5; the third line follows from equation (C.2), and $\sqrt{n}\mathbb{Z}_n(\boldsymbol{\alpha}_n) \Rightarrow N(0,1)$ under the local alternatives (i.e., Assumption 3.6(ii) under the alternatives).

For *Result (1)*, under local alternatives with $d_n = n^{-1/2}\|v_n^*\|_{\mathrm{sd}}$, we have

$$\mathcal{T}_n = -\big(\sqrt{n}\mathbb{Z}_n(\boldsymbol{\alpha}_n) - \kappa\big(1+o(1)\big)\big) \times \big(1 + o_{P_{n,Z^\infty}}(1)\big) + o_{P_{n,Z^\infty}}(1),$$

and

$$\mathcal{W}_n \equiv (\mathcal{T}_n)^2 \Rightarrow \chi_1^2\big(\kappa^2\big).$$

For *Result (2)*, under local alternatives with $\sqrt{n}\frac{d_n}{\|v_n^*\|_{\mathrm{sd}}} \to \infty$, we have

$$\begin{aligned}
\mathcal{W}_n &\equiv (\mathcal{T}_n)^2 \\
&= \left(O_{P_{n,Z^\infty}}(1) - \sqrt{n}\frac{d_n\kappa\big(1+o(1)\big)}{\|v_n^*\|_{\mathrm{sd}}}\right)^2 \times \big(1 + o_{P_{n,Z^\infty}}(1)\big) \\
&\quad + o_{P_{n,Z^\infty}}(1) \\
&\to \infty \quad \text{wpa1 } (P_{n,Z^\infty}).
\end{aligned}$$

Q.E.D.

PROOF OF THEOREM A.4: For *Result (1)*, following the proofs of Theorems 5.2(1) and A.2, we have: under local alternatives $\boldsymbol{\alpha}_n$ defined in (A.2), for $j = 1, 2$,

$$\widehat{W}_{j,n}^B = -\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)}{\sigma_\omega} + o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{n,Z^\infty}).$$

By similar calculations to those in the proof of Theorem 5.2(1), the law of $\sqrt{n}\frac{\mathbb{Z}_n^{\omega-1}(\boldsymbol{\alpha}_n)}{\sigma_\omega}$ is asymptotically (and wpa1 (P_{n,Z^∞})) equal to the law of $\mathbb{Z} \sim N(0,1)$. Then under the local alternatives $\boldsymbol{\alpha}_n$,

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\big|P_{V^\infty|Z^\infty}\big(\widehat{W}_{j,n}^B \le t|Z^n\big) - P_{Z^\infty}(\widehat{W}_n \le t)\big| \\
&\quad = o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{n,Z^\infty}),
\end{aligned} \tag{C.5}$$

where $\lim_{n\to\infty} P_{Z^\infty}(\widehat{W}_n \le t) = \Phi(t)$ (i.e., the standard normal c.d.f.). Thus the ath quantile of the distribution of $(\widehat{W}_{j,n}^B)^2$, $\widehat{c}_{j,n}(a)$, is uniformly bounded wpa1 (P_{n,Z^∞}).

For *Result (2a)*, by Theorem A.3(2), Result (1) (i.e., equation (C.5)) and the continuous mapping theorem, we have

$$\begin{aligned}
&P_{n,Z^\infty}\big(\mathcal{W}_n \ge \widehat{c}_{j,n}(1-\tau)\big) - P_{V^\infty|Z^\infty}\big(\big(\widehat{W}_{j,n}^B\big)^2 \ge \widehat{c}_{j,n}(1-\tau)|Z^n\big) \\
&\quad = \Pr\big(\chi_1^2\big(\kappa^2\big) \ge \widehat{c}_{j,n}(1-\tau)\big) - \Pr\big(\chi_1^2 \ge \widehat{c}_{j,n}(1-\tau)\big) \\
&\qquad + o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{n,Z^\infty}).
\end{aligned}$$

Thus, by the definition of $\widehat{c}_{j,n}(1-\tau)$, we obtain

$$\begin{aligned} &P_{n,Z^\infty}\big(\mathcal{W}_n \geq \widehat{c}_{j,n}(1-\tau)\big) \\ &\quad = \tau + \Pr\big(\chi_1^2\big(\kappa^2\big) \geq \widehat{c}_{j,n}(1-\tau)\big) - \Pr\big(\chi_1^2 \geq \widehat{c}_{j,n}(1-\tau)\big) \\ &\qquad + o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{n,Z^\infty}). \end{aligned}$$

Result (2b) directly follows from Theorem A.3(2), equation (C.5), and the continuous mapping theorem. *Q.E.D.*

C.3. *Proofs for Section* A.4 *on Asymptotic Theory Under Increasing Dimension of* ϕ

LEMMA C.1: *Let Assumption* 3.1(iv) *hold*. *Then*: *there exist positive finite constants* c, C *such that*

$$c^2 I_{d(n)} \leq \mathbb{D}_n^2 \leq C^2 I_{d(n)},$$

where $I_{d(n)}$ *is the* $d(n)\times d(n)$ *identity and for matrices* $A \leq B$ *means that* $B - A$ *is positive semi-definite*.

PROOF: By Assumption 3.1(iv), the eigenvalues of $\Sigma_0(x)$ and $\Sigma(x)$ are bounded away from zero and infinity uniformly in x. Therefore, for any matrix A,

$$A'\Sigma^{-1}(x)\Sigma_0(x)\Sigma^{-1}(x)A \geq d A'\Sigma^{-1}(x)A$$

and

$$A'\Sigma^{-1}(x)\Sigma_0(x)\Sigma^{-1}(x)A \leq D A'\Sigma^{-1}(x)A$$

for some finite constant $0 < d \leq D < \infty$, and for all x. Taking expectations at both sides and choosing $A' \equiv \frac{dm(x,\alpha_0)}{d\alpha}[\mathbf{v}_n^*]'$, these displays imply that

$$\Omega_{\mathrm{sd},n} \geq d\Omega_n \quad \text{and} \quad \Omega_{\mathrm{sd},n} \leq D\Omega_n.$$

Thus

$$\begin{aligned} \mathbb{D}_n^2 &= \Omega_{\mathrm{sd},n}^{1/2}\Omega_n^{-1}\Omega_{\mathrm{sd},n}\Omega_n^{-1}\Omega_{\mathrm{sd},n}^{1/2} \geq d\big\{\Omega_{\mathrm{sd},n}^{1/2}\Omega_n^{-1}\Omega_{\mathrm{sd},n}^{1/2}\big\} \\ &\geq d^2\Omega_{\mathrm{sd},n}^{1/2}\Omega_{\mathrm{sd},n}^{-1}\Omega_{\mathrm{sd},n}^{1/2} = d^2 I_{d(n)}. \end{aligned}$$

Similarly, $\mathbb{D}_n^2 \leq D^2 I_{d(n)}$. *Q.E.D.*

LEMMA C.2: *Let* $\mathcal{T}_n^M \equiv \{t \in \mathbb{R}^{d(n)} : \|t\|_e \le M_n n^{-1/2}\sqrt{d(n)}\}$. *Then*:

$$\big\|\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big\|_e = O_P\big(n^{-1/2}\sqrt{d(n)}\big) \quad \textit{and} \quad \Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n \in \mathcal{T}_n^M \quad \textit{wpa}1\,.$$

PROOF: Let $\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n \equiv n^{-1}\sum_{i=1}^n \zeta_{in}$ where $\zeta_{in} \in \mathbb{R}^{d(n)}$. Observe that $E[\zeta_{in}\zeta_{in}'] = I_{d(n)}$. It follows that

$$\begin{aligned} E_P\big[\big(\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big)'\big(\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big)\big] &= \mathrm{tr}\big\{E_P\big[\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\mathbf{Z}_n'\Omega_{\mathrm{sd},n}^{-1/2}\big]\big\} \\ &= n^{-2}\sum_{i=1}^n \mathrm{tr}\big\{E_P\big[\zeta_{in}\zeta_{in}'\big]\big\} = n^{-1}d(n), \end{aligned}$$

and thus the desired result follows by the Markov inequality. *Q.E.D.*

LEMMA C.3: *Let conditions for Lemma* 3.2 *and Assumption* A.3 *hold*. *Denote* $\widetilde{\gamma}_n \equiv \sqrt{s_n}(1+b_n)+a_n$. *Then*:

(1) $\|\Omega_{\mathrm{sd},n}^{-1/2}\{\mathbf{Z}_n + \langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\rangle\}\|_e = O_P(\sqrt{d(n)}\widetilde{\gamma}_n) = o_P(n^{-1/2})$;

(2) *further let Assumption* A.2 *hold*. *Then*

$$\big\|\Omega_{\mathrm{sd},n}^{-1/2}\big\{\mathbf{Z}_n + \phi(\widehat{\alpha}_n) - \phi(\alpha_0)\big\}\big\|_e = o_P\big(n^{-1/2}\big).$$

PROOF: For *Result (1)*, note that $\|t\|_e^2 = \sum_{l=1}^{d(n)} |t_l|^2$ and if we obtain $|t_l| = O_P(\widetilde{\gamma}_n)$ for $\widetilde{\gamma}_n$ uniformly over l, then $\|t\|_e^2 = O_P(d(n)\widetilde{\gamma}_n^2)$.

The rest of the proof follows closely the proof of Theorem 4.1, so we only present the main steps. By definition of the approximate PSMD estimator $\widehat{\alpha}_n$, and Assumption A.3(i),

$$0 \le t'\Omega_{\mathrm{sd},n}^{-1/2}\big(\mathbf{Z}_n + \big\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\big\rangle\big) + \frac{1}{2}t'\mathbb{B}_n t + O_P\big(r^{-1}(t)\big).$$

We now choose $t = \sqrt{s_n}e$, where $e \in \{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,1)\}$; it is easy to see that this $t \in \mathcal{T}_n^M$, and thus the display above implies

$$0 \le e'\Omega_{\mathrm{sd},n}^{-1/2}\big(\mathbf{Z}_n + \big\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\big\rangle\big) + O_P(\widetilde{\gamma}_n).$$

By changing the sign of t, it follows that

$$\big|e'\Omega_{\mathrm{sd},n}^{-1/2}\big(\mathbf{Z}_n + \big\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\big\rangle\big)\big| = O_P(\widetilde{\gamma}_n).$$

Observe that the RHS holds uniformly over e, thus, since $e \in \{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,1)\}$, it follows that

$$\big\|\Omega_{\mathrm{sd},n}^{-1/2}\big(\mathbf{Z}_n + \big\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\big\rangle\big)\big\|_e = O_P\big(\sqrt{d(n)}\widetilde{\gamma}_n\big) = o_P\big(n^{-1/2}\big),$$

where the second equal sign is due to Assumption A.3(ii).

For *Result (2)*. In view of Result (1), it suffices to show that

$$\left\|\Omega_{\mathrm{sd},n}^{-1/2}\left\{\phi(\widehat{\alpha}_n)-\phi(\alpha_0)-\left\langle \mathbf{v}_n^{*\prime},\widehat{\alpha}_n-\alpha_0\right\rangle\right\}\right\|_e=o_P\left(n^{-1/2}\right).$$

Following the proof of Theorem 4.1, we have

$$\begin{aligned}\left\langle \mathbf{v}_n^{*\prime},\widehat{\alpha}_n-\alpha_0\right\rangle&=\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n-\alpha_{0,n}]\\&=\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n-\alpha_0]-\frac{d\phi(\alpha_0)}{d\alpha}[\alpha_{0,n}-\alpha_0].\end{aligned}$$

Since Assumption A.2(ii)(iii) (with $t=0$) implies that

$$\begin{aligned}&\left\|\Omega_{\mathrm{sd},n}^{-1/2}\left\{\phi(\widehat{\alpha}_n)-\phi(\alpha_0)-\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n-\alpha_0]+\frac{d\phi(\alpha_0)}{d\alpha}[\alpha_{0,n}-\alpha_0]\right\}\right\|_e\\&\quad=O_P(c_n),\end{aligned}$$

the desired result now follows from Assumption A.2(iv) of $c_n=o(n^{-1/2})$. *Q.E.D.*

PROOF OF THEOREM A.5: Throughout the proof, let $\widehat{W}_n\equiv n(\phi(\widehat{\alpha}_n)-\phi(\alpha_0))'\Omega_{\mathrm{sd},n}^{-1}(\phi(\widehat{\alpha}_n)-\phi(\alpha_0))$. By Lemma C.3(2),

$$T_n\equiv\left(\phi(\widehat{\alpha}_n)-\phi(\alpha_0)+\mathbf{Z}_n\right)'\Omega_{\mathrm{sd},n}^{-1}\left(\phi(\widehat{\alpha}_n)-\phi(\alpha_0)+\mathbf{Z}_n\right)=o_P\left(n^{-1}\right).$$

Observe that

$$\begin{aligned}&\left|\left(\phi(\widehat{\alpha}_n)-\phi(\alpha_0)\right)'\Omega_{\mathrm{sd},n}^{-1}\left(\phi(\widehat{\alpha}_n)-\phi(\alpha_0)\right)-(\mathbf{Z}_n)'\Omega_{\mathrm{sd},n}^{-1}(\mathbf{Z}_n)\right|\\&\quad\le T_n+2\left\|\left(\phi(\widehat{\alpha}_n)-\phi(\alpha_0)+\mathbf{Z}_n\right)'\Omega_{\mathrm{sd},n}^{-1/2}\right\|_e\times\left\|\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\right\|_e\\&\quad=o_P\left(n^{-1}\right)+2\left\|\left(\phi(\widehat{\alpha}_n)-\phi(\alpha_0)+\mathbf{Z}_n\right)'\Omega_{\mathrm{sd},n}^{-1/2}\right\|_e\times\left\|\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\right\|_e\\&\quad=o_P\left(n^{-1}\right)+o_P\left(n^{-1}\sqrt{d(n)}\right),\end{aligned}$$

where the last equality is due to Lemmas C.2 and C.3(2). Therefore we obtain *Result (1)*:

$$\widehat{W}_n=(\sqrt{n}\mathbf{Z}_n)'\Omega_{\mathrm{sd},n}^{-1}(\sqrt{n}\mathbf{Z}_n)+o_P\left(\sqrt{d(n)}\right)\equiv\mathbf{W}_n+o_P\left(\sqrt{d(n)}\right).$$

Result (2) follows directly from Result (1) when $d(n)=d$ is fixed and finite.

Result (3) follows from Result (1) and the following property:

$$\Xi_n\equiv\left(2d(n)\right)^{-1/2}\left(\mathbf{W}_n-d(n)\right)\Rightarrow N(0,1)$$

where $\mathbf{W}_n \equiv (\sqrt{n}\mathbf{Z}_n)'\Omega_{\mathrm{sd},n}^{-1}(\sqrt{n}\mathbf{Z}_n)$, or formally,

$$\sup_{f\in BL_1(\mathbb{R})}\big|E\big[f(\Xi_n)\big]-E\big[f(\mathbb{Z})\big]\big|=o(1)$$

where $\mathbb{Z}\sim N(0,1)$ and $BL_1(\mathbb{R})$ is the space of bounded (by 1) Lipschitz functions from $\mathbb{R}$ to $\mathbb{R}$.

By the triangle inequality, it suffices to show that

$$\sup_{f\in BL_1(\mathbb{R})}\big|E\big[f(\Xi_n)\big]-E\big[f(\xi_n)\big]\big|=o(1) \tag{C.6}$$

and

$$\sup_{f\in BL_1(\mathbb{R})}\big|E\big[f(\xi_n)\big]-E\big[f(\mathbb{Z})\big]\big|=o(1), \tag{C.7}$$

where $\xi_n \equiv (2d(n))^{-1/2}(\sum_{j=1}^{d(n)}\mathbb{Z}_j^2 - d(n))$ with $\mathbb{Z}_j\sim N(0,1)$ and independent across $j=1,\dots,d(n)$. We now show that both equations hold.

Equation (C.6). Let $t\mapsto \nu_M(t)\equiv \min\{t't, M\}$ for some $M>0$. Observe

$$\begin{aligned}
&\big|E\big[f(\Xi_n)\big]-E\big[f\big((2d(n))^{-1/2}\big(\nu_M\big(\Omega_{\mathrm{sd},n}^{-1/2}\sqrt{n}\mathbf{Z}_n\big)-d(n)\big)\big)\big]\big|\\
&\quad=\big|E\big[f\big((2d(n))^{-1/2}\big(\nu_\infty\big(\Omega_{\mathrm{sd},n}^{-1/2}\sqrt{n}\mathbf{Z}_n\big)-d(n)\big)\big)\\
&\qquad-f\big((2d(n))^{-1/2}\big(\nu_M\big(\Omega_{\mathrm{sd},n}^{-1/2}\sqrt{n}\mathbf{Z}_n\big)-d(n)\big)\big)\big]\big|\\
&\quad=\bigg|\int_{\{z:nz'\Omega_{\mathrm{sd},n}^{-1}z>M\}}\bigg[f\bigg(\frac{\nu_\infty\big(\Omega_{\mathrm{sd},n}^{-1/2}\sqrt{n}z_n\big)-d(n)}{\sqrt{2d(n)}}\bigg)\\
&\qquad-f\bigg(\frac{M-d(n)}{\sqrt{2d(n)}}\bigg)\bigg]P_{Z^\infty}(dz)\bigg|\\
&\quad\le 2P_{Z^\infty}\big((\sqrt{n}\mathbf{Z}_n)'\Omega_{\mathrm{sd},n}^{-1}(\sqrt{n}\mathbf{Z}_n)>M\big),
\end{aligned}$$

where the last line follows from the fact that f is bounded by 1. Therefore, by the Markov inequality, for any ϵ, there exists an M such that

$$\big|E\big[f(\Xi_n)\big]-E\big[f\big((2d(n))^{-1/2}\big(\nu_M\big(\Omega_{\mathrm{sd},n}^{-1/2}\sqrt{n}\mathbf{Z}_n\big)-d(n)\big)\big)\big]\big|<\epsilon$$

for sufficiently large n. A similar result holds if we replace $\Omega_{\mathrm{sd},n}^{-1/2}\sqrt{n}\mathbf{Z}_n$ by $\mathcal{Z}_n=(\mathbb{Z}_1,\dots,\mathbb{Z}_{d(n)})'$ with $\mathbb{Z}_j\sim N(0,1)$ and independent across $j=1,\dots,d(n)$.

Therefore, in order to show equation (C.6), it suffices to show

$$\sup_{f\in BL_1(\mathbb{R})} \left|E\left[f(\Xi_{M,n})\right]-E\left[f(\xi_{M,n})\right]\right| = o(1),$$

where $\Xi_{M,n} \equiv (2d(n))^{-1/2}(\nu_M(\Omega_{\mathrm{sd},n}^{-1/2}\sqrt{n}\mathbf{Z}_n) - d(n))$ and $\xi_{M,n} \equiv (2d(n))^{-1/2} \times (\nu_M(\mathcal{Z}_n) - d(n))$.

Since f is uniformly bounded and continuous, it is clear that in order to show the previous display, it suffices to show that

$$\text{(C.8)} \qquad \left(2d(n)\right)^{-1/2}\left|\nu_M\left(\Omega_{\mathrm{sd},n}^{-1/2}\sqrt{n}\mathbf{Z}_n\right) - \nu_M(\mathcal{Z}_n)\right| = o_P(1).$$

It turns out that $|\nu_M(t)-\nu_M(r)| \le 2\sqrt{M}\|t-r\|_e$, so $t\mapsto \nu_M(t)$ is Lipschitz (and uniformly bounded). So in order to show equation (C.8), it is sufficient to show that for any $\delta>0$, there exists an $N(\delta)$ such that

$$\Pr\left(\left(2d(n)\right)^{-1/2}\left\|\Omega_{\mathrm{sd},n}^{-1/2}\sqrt{n}\mathbf{Z}_n - \mathcal{Z}_n\right\|_e > \delta\right) < \delta$$

for all $n \ge N(\delta)$. Note that $\Omega_{\mathrm{sd},n}^{-1/2}\sqrt{n}\mathbf{Z}_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n \Psi_n(Z_i)$, with $\Psi_n(z) \equiv (\frac{dm(x,\alpha_0)}{d\alpha}[\mathbf{v}_n^*]\Omega_{\mathrm{sd},n}^{-1/2})'\rho(z,\alpha_0)$, and that $\mathcal{Z}_n$ can be cast as $\frac{1}{\sqrt{n}}\sum_{i=1}^n \mathcal{Z}_{n,i}$ with $\mathcal{Z}_{n,i} \sim N(0, I_{d(n)})$, i.i.d. across $i=1,\dots,n$. Following the arguments in Section 10.4 of Pollard (2002), we obtain: for any $\delta>0$,

$$\Pr\left(\left\|\sqrt{n}\mathbf{Z}_n'\Omega_{\mathrm{sd},n}^{-1/2} - \frac{1}{\sqrt{n}}\sum_{i=1}^n \mathcal{Z}_{n,i}\right\|_e > 3\delta\right) \le \mathcal{Y}_{d(n)}\left(\frac{\mu_{3,n}nd(n)^{5/2}}{(\delta\sqrt{n})^3}\right),$$

for any n, where $x\mapsto \mathcal{Y}_{d(n)}(x) \equiv Cx \times (1+|\log(1/x)|/d(n))$ and $\mu_{3,n} \equiv E[\|(\frac{dm(X,\alpha_0)}{d\alpha}[\mathbf{v}_n^*]\Omega_{\mathrm{sd},n}^{-1/2})'\rho(Z,\alpha_0)\|_e^3]$. Therefore,

$$\begin{aligned}
&\Pr\left(\left(2d(n)\right)^{-1/2}\left\|\Omega_{\mathrm{sd},n}^{-1/2}\sqrt{n}\mathbf{Z}_n - \mathcal{Z}_n\right\|_e > \delta\right) \\
&\quad \le \mathcal{Y}_{d(n)}\left(\frac{\mu_{3,n}nd(n)^{5/2}}{(\delta/3)^3 d(n)^{3/2}n^{3/2}}\right) = \mathcal{Y}_{d(n)}\left(n^{-1/2}d(n)\frac{\mu_{3,n}}{(\delta/3)^3 8}\right) \to 0
\end{aligned}$$

provided that $d(n) = o(\sqrt{n}\mu_{3,n}^{-1})$ which is assumed in the Theorem Result (3).

Equation (C.7). Observe that $\xi_n \equiv (2d(n))^{-1/2}(\sum_{j=1}^{d(n)}\mathbb{Z}_j^2 - d(n))$ with $\mathbb{Z}_j \sim N(0,1)$ i.i.d. across $j=1,\dots,d(n)$, $E[(\mathbb{Z}_l^2-1)] = 0$, and $E[(\mathbb{Z}_l^2-1)^2] = 2$. Thus, $\xi_n \Rightarrow N(0,1)$ by a standard CLT. *Q.E.D.*

In the following, we recall that $\alpha(t) \equiv \alpha + \mathbf{v}_n^*(\Omega_{\mathrm{sd},n})^{-1/2}t$ for $t\in\mathbb{R}^{d(n)}$.

LEMMA C.4: *Let all conditions for Theorem* A.6(1) *hold*. *Then there exists a* t_n (*possibly random*) *such that*: (1) $t_n \in \mathcal{T}_n^M$ *wpa*1, (2) $\widehat{\alpha}_n(t_n) \in \mathcal{A}_{k(n)}^R = \{\alpha\in\mathcal{A}_{k(n)}:$

$\phi(\alpha) = \phi_0\}$ *wpa*1, *and* (3)

$$0 \le n\{\widehat{Q}_n(\widehat{\alpha}_n(t_n)) - \widehat{Q}_n(\widehat{\alpha}_n)\}$$
$$\le (\sqrt{n}\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n)'\mathbb{D}_n(\sqrt{n}\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n) + o_P(\sqrt{d(n)}).$$

PROOF: To show *Parts (1) and (2)*, we define the following mappings:

$$t \in \mathbb{R}^{d(n)} \mapsto \varphi_n(t) \equiv \Omega_{\mathrm{sd},n}^{-1/2}\left\{\phi(\widehat{\alpha}_n(t)) - \phi(\alpha_0) - \frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n(t) - \alpha_0]\right\}$$

and $t \in \mathbb{R}^{d(n)} \mapsto \tau_n(t) \equiv -\mathbb{D}_n\Omega_{\mathrm{sd},n}^{-1/2}\{\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\rangle + \frac{d\phi(\alpha_0)}{d\alpha}[\alpha_{0n} - \alpha_0] + \Omega_{\mathrm{sd},n}^{1/2}t\}$. Under our assumptions, both mappings are continuous in t (a.s.) and thus $\Phi_n \equiv \varphi_n \circ \tau_n$ is also continuous in t (a.s.). Given $c_n = o(n^{-1/2})$ satisfying Assumption A.2(iv), we define $\mathbb{T}_n \equiv \{t \in \mathbb{R}^{d(n)} : \|t\|_e \le L_n c_n\}$ where $(L_n)_n$ is a positive real-valued sequence diverging to infinity slowly such that $L_n c_n = o(n^{-1/2})$ (such a sequence exists by Assumption A.2(iv)).

Let $S_n \equiv \{Z^n : \sup_{t\in\mathbb{T}_n} \|\Phi_n(t)\|_e \le L_n c_n\}$. By Lemmas C.1, C.2, and C.3(2), and Assumption A.2(iii), we have that, for any $t \in \mathbb{T}_n$,

$$\|\tau_n(t)\|_e \le O_P(\sqrt{d(n)}\{\widetilde{\gamma}_n + n^{-1/2}\}) + O(c_n) + \|\mathbb{D}_n t\|_e$$
$$= O_P(n^{-1/2}\sqrt{d(n)}) + O(L_n c_n),$$

where $\widetilde{\gamma}_n \equiv \sqrt{|s_n|}(1 + b_n) + a_n = o(n^{-1/2})$ (by Assumption A.3(ii)). Hence $\tau_n(t) \in \mathcal{T}_n^M$ for all $t \in \mathbb{T}_n$. This implies, by Assumption A.2(i)(ii), $P(S_n) \to 1$.

Moreover, these results imply that $\|\Phi_n(t)\|_e \le L_n c_n$ for all $t \in \mathbb{T}_n$ and $Z^n \in S_n$. This implies that $\{\Phi_n(t) : t \in \mathbb{T}_n\} \subseteq \mathbb{T}_n$ wpa1.

For any given n, $\mathbb{T}_n$ is compact and convex in $\mathbb{R}^{d(n)}$ and since Φ_n is continuous and maps $\mathbb{T}_n$ into itself (wpa1), by the Brouwer fixed point theorem, wpa1 there exists a $\widehat{t}_n \in \mathbb{T}_n$ such that $\Phi_n(\widehat{t}_n) = \widehat{t}_n$. Therefore,

$$\widehat{t}_n = \varphi_n \circ \tau_n(\widehat{t}_n)$$
$$= \Omega_{\mathrm{sd},n}^{-1/2}\left\{\phi(\widehat{\alpha}_n(\tau_n(\widehat{t}_n))) - \phi(\alpha_0) - \frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n(\tau_n(\widehat{t}_n)) - \alpha_0]\right\}.$$

Since

$$\widehat{\alpha}_n(\tau_n(\widehat{t}_n)) = \widehat{\alpha}_n + \mathbf{v}_n^*\Omega_{\mathrm{sd},n}^{-1/2}\tau_n(\widehat{t}_n)$$
$$= \widehat{\alpha}_n - \mathbf{v}_n^*\Omega_{\mathrm{sd},n}^{-1/2}\mathbb{D}_n\Omega_{\mathrm{sd},n}^{-1/2}\Big(\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\rangle$$
$$+ \frac{d\phi(\alpha_0)}{d\alpha}[\alpha_{0n} - \alpha_0] + \Omega_{\mathrm{sd},n}^{1/2}\widehat{t}_n\Big)$$

and $\Omega_{\mathrm{sd},n}^{-1/2}\mathbb{D}_n\Omega_{\mathrm{sd},n}^{-1/2}=\Omega_n^{-1}$, we obtain

$$\begin{aligned}
&\frac{d\phi(\alpha_0)}{d\alpha}\big[\widehat{\alpha}_n\big(\tau_n(\widehat{t}_n)\big)-\alpha_0\big]\\
&\quad=\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n-\alpha_0]\\
&\qquad-\big\langle\mathbf{v}_n^{*\prime},\mathbf{v}_n^*\big\rangle\Omega_n^{-1}\bigg(\big\langle\mathbf{v}_n^{*\prime},\widehat{\alpha}_n-\alpha_0\big\rangle+\frac{d\phi(\alpha_0)}{d\alpha}[\alpha_{0n}-\alpha_0]+\Omega_{\mathrm{sd},n}^{1/2}\widehat{t}_n\bigg).
\end{aligned}$$

Since $\langle\mathbf{v}_n^{*\prime},\mathbf{v}_n^*\rangle=\Omega_n$, we obtain

$$\begin{aligned}
\widehat{t}_n&=\Omega_{\mathrm{sd},n}^{-1/2}\bigg\{\phi\big(\widehat{\alpha}_n\big(\tau_n(\widehat{t}_n)\big)\big)-\phi(\alpha_0)-\frac{d\phi(\alpha_0)}{d\alpha}[\widehat{\alpha}_n-\alpha_0]\\
&\qquad+\big\langle\mathbf{v}_n^{*\prime},\widehat{\alpha}_n-\alpha_0\big\rangle+\frac{d\phi(\alpha_0)}{d\alpha}[\alpha_{0n}-\alpha_0]+\Omega_{\mathrm{sd},n}^{1/2}\widehat{t}_n\bigg\}\\
&=\Omega_{\mathrm{sd},n}^{-1/2}\big\{\phi\big(\widehat{\alpha}_n\big(\tau_n(\widehat{t}_n)\big)\big)-\phi(\alpha_0)\big\}+\widehat{t}_n.
\end{aligned}$$

Thus $\Omega_{\mathrm{sd},n}^{-1/2}\{\phi(\widehat{\alpha}_n(\tau_n(\widehat{t}_n)))-\phi(\alpha_0)\}=0$ wpa1 iff $\phi(\widehat{\alpha}_n(\tau_n(\widehat{t}_n)))-\phi(\alpha_0)=0$ wpa1. Also, since $\tau_n(\widehat{t}_n)\in\mathcal{T}_n^M$ wpa1, Parts (1) and (2) hold with $t_n\equiv\tau_n(\widehat{t}_n)$.

To show *Part (3)*, recall that $\widehat{\alpha}_n\in\mathcal{N}_{osn}$ wpa1 and $\widehat{\alpha}_n(t_n)\in\mathcal{A}_{k(n)}^R$ by Parts (1) and (2) with $t_n\equiv\tau_n(\widehat{t}_n)$. We can rewrite t_n as

$$\begin{aligned}
t_n&\equiv-\mathbb{D}_n\Omega_{\mathrm{sd},n}^{-1/2}\big\{\big\langle\mathbf{v}_n^{*\prime},\widehat{\alpha}_n-\alpha_0\big\rangle+A_n(\widehat{t}_n)\big\}\quad\text{with}\\
A_n(t)&\equiv\frac{d\phi(\alpha_0)}{d\alpha}[\alpha_{0n}-\alpha_0]+\Omega_{\mathrm{sd},n}^{1/2}t.
\end{aligned}$$

Observe that $\|t_n\|_e=O_P(\sqrt{d(n)}n^{-1/2})$, so by Assumption A.3(i) and the definition of $\widehat{\alpha}_n$,

$$\begin{aligned}
0&\le n\big[\widehat{Q}_n\big(\widehat{\alpha}_n(t_n)\big)-\widehat{Q}_n(\widehat{\alpha}_n)\big]\\
&=n(t_n)'\Omega_{\mathrm{sd},n}^{-1/2}\big\{\big\langle\mathbf{v}_n^{*\prime},\widehat{\alpha}_n-\alpha_0\big\rangle+\mathbf{Z}_n\big\}\\
&\qquad+0.5n\big\{t_n'\mathbb{B}_nt_n\big\}+n\times O_P\big(s_n+\|t_n\|_ea_n+\|t_n\|_e^2b_n\big)\\
&\le n(t_n)'\Omega_{\mathrm{sd},n}^{-1/2}\big\{\big\langle\mathbf{v}_n^{*\prime},\widehat{\alpha}_n-\alpha_0\big\rangle+\mathbf{Z}_n\big\}\\
&\qquad+0.5n\big\{t_n'\mathbb{D}_n^{-1}t_n\big\}+n\times O_P\big(s_n+\|t_n\|_ea_n+\|t_n\|_e^2b_n\big),
\end{aligned}$$

where the third line follows from the fact that $\sup_{t:\|t\|_e=1}|t'\{\mathbb{B}_n-\mathbb{D}_n^{-1}\}t|=O_P(b_n)$ by assumption, and thus we have: $t'\mathbb{B}_nt\le|t'\{\mathbb{B}_n-\mathbb{D}_n^{-1}\}t|+t'\mathbb{D}_n^{-1}t\le\|t\|_e^2O_P(b_n)+t'\mathbb{D}_n^{-1}t$ uniformly over $t\in\mathbb{R}^{d(n)}$ with $\|t\|_e=1$.

By the fact that $\Omega_{\mathrm{sd},n}^{-1/2}\mathbb{D}_n\mathbb{D}_n^{-1}\mathbb{D}_n\Omega_{\mathrm{sd},n}^{-1/2} = \Omega_{\mathrm{sd},n}^{-1/2}\mathbb{D}_n\Omega_{\mathrm{sd},n}^{-1/2} = \Omega_n^{-1}$, the definition of t_n, and straightforward algebra, the previous display implies that

$$\begin{aligned}
& n\big[\widehat{Q}_n\big(\widehat{\alpha}_n(t_n)\big) - \widehat{Q}_n(\widehat{\alpha}_n)\big] \\
&\quad \leq -0.5n\big(\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\rangle\big)'\Omega_n^{-1}\big(\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\rangle\big) \\
&\qquad - n\big(\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\rangle\big)'\Omega_n^{-1}(\mathbf{Z}_n) - n\big(A_n(\widehat{t}_n)\big)'\Omega_n^{-1}\mathbf{Z}_n \\
&\qquad + 0.5n\big(A_n(\widehat{t}_n)\big)'\Omega_n^{-1}\big(A_n(\widehat{t}_n)\big) + n \times O_P\big(s_n + \|t_n\|_e a_n + \|t_n\|_e^2 b_n\big) \\
&\quad \leq n(\mathbf{Z}_n)'\Omega_n^{-1}(\mathbf{Z}_n) - n\big(\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\rangle + \mathbf{Z}_n\big)'\Omega_n^{-1}(\mathbf{Z}_n) \\
&\qquad - n\big(A_n(\widehat{t}_n)\big)'\Omega_n^{-1}\mathbf{Z}_n + 0.5n\big(A_n(\widehat{t}_n)\big)'\Omega_n^{-1}\big(A_n(\widehat{t}_n)\big) \\
&\qquad + n \times O_P\big(s_n + \|t_n\|_e a_n + \|t_n\|_e^2 b_n\big),
\end{aligned}$$

where the second line follows by $(\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\rangle)'\Omega_n^{-1}(\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\rangle) \geq 0$ and straightforward algebra. Observe that

$$\begin{aligned}
& \big[\big(A_n(\widehat{t}_n)\big)'\Omega_n^{-1}\big(A_n(\widehat{t}_n)\big)\big]^{1/2} \\
&\quad = \big[\big(A_n(\widehat{t}_n)\big)'\Omega_{\mathrm{sd},n}^{-1/2}\mathbb{D}_n\Omega_{\mathrm{sd},n}^{-1/2}\big(A_n(\widehat{t}_n)\big)\big]^{1/2} \\
&\quad = O_P\left(\left\|\mathbb{D}_n^{1/2}\Omega_{\mathrm{sd},n}^{-1/2}\frac{d\phi(\alpha_0)}{d\alpha}[\alpha_{0n} - \alpha_0]\right\|_e + \big\|\mathbb{D}_n^{1/2}\widehat{t}_n\big\|_e\right) \\
&\quad = O_P\left(\left\|\Omega_{\mathrm{sd},n}^{-1/2}\frac{d\phi(\alpha_0)}{d\alpha}[\alpha_{0n} - \alpha_0]\right\|_e + \|\widehat{t}_n\|_e\right) \\
&\quad = O_P\big(c_n(1 + L_n)\big) = o_P\big(n^{-1/2}\big),
\end{aligned}$$

where the first equation follows from Lemma C.1, and the last equality follows from Assumption A.2(iii) and the results from Parts (1) and (2). Also

$$\big[(\mathbf{Z}_n)'\Omega_n^{-1}(\mathbf{Z}_n)\big]^{1/2} = \big[\big(\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big)'\mathbb{D}_n\big(\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big)\big]^{1/2} = O_P\big(n^{-1/2}\sqrt{d(n)}\big)$$

by Lemmas C.1 and C.2. Thus $(A_n(\widehat{t}_n))'\Omega_n^{-1}\mathbf{Z}_n = o_P(n^{-1/2})O_P(n^{-1/2}\sqrt{d(n)}) = o_P(n^{-1}\sqrt{d(n)})$. Similarly, $(\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n - \alpha_0\rangle + \mathbf{Z}_n)'\Omega_n^{-1}(\mathbf{Z}_n) = o_P(n^{-1}\sqrt{d(n)})$ by Lemmas C.1 and C.3(1). Since $\|t_n\|_e = O_P(\sqrt{d(n)}n^{-1/2})$, we have $n(s_n + \|t_n\|_e a_n + (\|t_n\|_e^2)b_n) = O_P(ns_n + \sqrt{d(n)}n^{1/2}a_n + \sqrt{d(n)}\sqrt{d(n)}b_n) = o_P(n^{-1} \times \sqrt{d(n)})$ under Assumption A.3(ii). Therefore,

$$\begin{aligned}
& n\big[\widehat{Q}_n\big(\widehat{\alpha}_n(t_n)\big) - \widehat{Q}_n(\widehat{\alpha}_n)\big] \\
&\quad \leq n(\mathbf{Z}_n)'\Omega_n^{-1}(\mathbf{Z}_n) + o_P\big(\sqrt{d(n)}\big) \\
&\quad = \big(\sqrt{n}\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big)'\mathbb{D}_n\big(\sqrt{n}\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big) + o_P\big(\sqrt{d(n)}\big).
\end{aligned}$$

Q.E.D.

PROOF OF THEOREM A.6: The proof is very similar to that of Theorem 4.3 and we only provide main steps here.

STEP 1: Similarly to Steps 1 and 2 in the proof of Theorem 4.3, by the definitions of $\widehat{\alpha}_n^R$ and $\widehat{\alpha}_n$ and Assumption A.3(i), it follows that for any (possibly random) $t \in \mathcal{T}_n^M$,

$$\begin{aligned} 0.5\,\widehat{\mathrm{QLR}}_n(\phi_0) &\geq 0.5n\big(\widehat{Q}_n\big(\widehat{\alpha}_n^R\big) - \widehat{Q}_n\big(\widehat{\alpha}_n^R(t)\big)\big) - o_P(1) \\ &= -n\big(t'\Omega_{\mathrm{sd},n}^{-1/2}\big\{\mathbf{Z}_n + \big\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n^R - \alpha_0\big\rangle\big\} + 0.5t'\mathbb{B}_n t\big) \\ &\quad + O_P\big(s_n n + n\|t\|_e a_n + n\|t\|_e^2 b_n\big). \end{aligned}$$

By Assumption A.2(i)(ii),

$$\Bigg\| \Omega_{\mathrm{sd},n}^{-1/2}\bigg(\underbrace{\phi\big(\widehat{\alpha}_n^R\big) - \phi(\alpha_0)}_{=0} - \frac{d\phi(\alpha_0)}{d\alpha}\big[\widehat{\alpha}_n^R - \alpha_0\big]\bigg)\Bigg\|_e = O_P(c_n).$$

Hence, by Assumption A.2(iii),

$$\big\|\Omega_{\mathrm{sd},n}^{-1/2}\big\langle \mathbf{v}_n^{*\prime}, \widehat{\alpha}_n^R - \alpha_0\big\rangle\big\|_e = O_P(c_n). \tag{C.9}$$

Since $\sup_{t:\|t\|_e=1}|t'\{\mathbb{B}_n - \mathbb{D}_n^{-1}\}t| = O_P(b_n)$ by assumption, we have: $t'\mathbb{B}_n t \leq |t'\{\mathbb{B}_n - \mathbb{D}_n^{-1}\}t| + t'\mathbb{D}_n^{-1}t \leq \|t\|_e^2 O_P(b_n) + t'\mathbb{D}_n^{-1}t$ uniformly over $t \in \mathbb{R}^{d(n)}$ with $\|t\|_e = 1$. This, Assumption A.3(i), and equation (C.9) together imply that

$$\begin{aligned} 0.5\,\widehat{\mathrm{QLR}}_n(\phi_0) &\geq -n\big(t'\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n + 0.5t'\mathbb{D}_n^{-1}t\big) \\ &\quad + O_P\big(s_n n + n\|t\|_e(a_n + c_n) + n\|t\|_e^2 b_n\big). \end{aligned}$$

In the above display, we let $t' = -\mathbf{Z}_n'\Omega_{\mathrm{sd},n}^{-1/2}\mathbb{D}_n$, which, by Lemmas C.1 and C.2, is an admissible choice and $\|t\|_e = O_P(n^{-1/2}\sqrt{d(n)})$. Observe that $t_n'\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n = -\mathbf{Z}_n'\Omega_{\mathrm{sd},n}^{-1/2}\mathbb{D}_n\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n$ and $t_n'\mathbb{D}_n^{-1}t_n = \mathbf{Z}_n'\Omega_{\mathrm{sd},n}^{-1/2}\mathbb{D}_n\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n$; we obtain

$$\begin{aligned} 0.5\,\widehat{\mathrm{QLR}}_n(\phi_0) &\geq 0.5\big(\sqrt{n}\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big)'\mathbb{D}_n\big(\sqrt{n}\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big) \\ &\quad + O_P\big(s_n n + n^{1/2}\sqrt{d(n)}(a_n + c_n) + d(n)b_n\big) \\ &= 0.5\big(\sqrt{n}\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big)'\mathbb{D}_n\big(\sqrt{n}\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big) + o_P\big(\sqrt{d(n)}\big), \end{aligned}$$

where the last equal sign is due to Assumptions A.2(iv) and A.3(ii).

STEP 2: Similarly to Step 3 in the proof of Theorem 4.3, by the definitions of $\widehat{\alpha}_n^R$ and $\widehat{\alpha}_n$ and the result that $\widehat{\alpha}_n(t_n) \in \mathcal{A}_{k(n)}^R$ (Lemma C.4), with t_n given in Lemma C.4, we obtain

$$0.5\,\widehat{\mathrm{QLR}}_n(\phi_0) \leq 0.5n\big(\widehat{Q}_n\big(\widehat{\alpha}_n(t_n)\big) - \widehat{Q}_n(\widehat{\alpha}_n)\big) + o_P(1).$$

By Lemma C.4(3), it follows that

$$0.5\,\widehat{\mathrm{QLR}}_n(\phi_0) \leq 0.5\big(\sqrt{n}\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big)'\mathbb{D}_n\big(\sqrt{n}\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big) + o_P\big(\sqrt{d(n)}\big).$$

STEP 3: The results in Steps 1 and 2 together imply that

$$\widehat{\mathrm{QLR}}_n(\phi_0) = \big(\sqrt{n}\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big)'\mathbb{D}_n\big(\sqrt{n}\Omega_{\mathrm{sd},n}^{-1/2}\mathbf{Z}_n\big) + o_P\big(\sqrt{d(n)}\big),$$

which establishes *Result (1)*.

Result (2) directly follows from Result (1) and the fact that $\mathbb{D}_n = I_{d(n)}$, $\Omega_{\mathrm{sd},n} = \Omega_{0,n}$ when $\Sigma = \Sigma_0$.

Result (3) follows from Result (2), $\Omega_{\mathrm{sd},n} = \Omega_{0,n}$ when $\Sigma = \Sigma_0$, and the following property of $\mathbf{W}_n \equiv n\mathbf{Z}_n'\Omega_{\mathrm{sd},n}^{-1}\mathbf{Z}_n$:

$$\big(2d(n)\big)^{-1/2}\big(\mathbf{W}_n - d(n)\big) \Rightarrow N(0,1),$$

which has been established in the proof of Theorem A.5 Result (3). *Q.E.D.*

C.4. *Proofs for Section* A.6 *on Series LS Estimator* $\widehat{m}$ *and Its Bootstrap Version*

PROOF OF LEMMA A.2: For *Result (1)*, since

$$\begin{aligned} M_n\big(Z^n\big) &\equiv P_{V^\infty|Z^\infty}\Bigg(\sup_{\mathcal{N}_{osn}} \frac{\overline{\tau}_n}{n}\sum_{i=1}^n \big\|\widehat{m}^B(X_i,\alpha) - \widetilde{m}(X_i,\alpha) \\ &\qquad - \widehat{m}^B(X_i,\alpha_0)\big\|_e^2 \geq M \Big| Z^n\Bigg) \\ &\leq P_{V^\infty|Z^\infty}\Bigg(\sup_{\mathcal{N}_{osn}} \frac{\overline{\tau}_n}{n}\sum_{i=1}^n \big\|\widehat{m}^B(X_i,\alpha) - \widehat{m}(X_i,\alpha) \\ &\qquad - \big\{\widehat{m}^B(X_i,\alpha_0) - \widehat{m}(X_i,\alpha_0)\big\}\big\|_e^2 \geq \frac{M}{2} \Big| Z^n\Bigg) \\ &\quad + P_{V^\infty|Z^\infty}\Bigg(\sup_{\mathcal{N}_{osn}} \frac{\overline{\tau}_n}{n}\sum_{i=1}^n \big\|\widehat{m}(X_i,\alpha) - \widetilde{m}(X_i,\alpha) \end{aligned}$$

$$- \widehat{m}(X_i, \alpha_0)\big\|_e^2 \geq \frac{M}{2}\Big| Z^n \Big) \equiv M_{1,n}\big(Z^n\big) + M_{2,n}\big(Z^n\big),$$

we have: for all $\delta > 0$, there is an $M(\delta) > 0$ such that, for all $M \geq M(\delta)$,

$$P_{Z^\infty}\big(M_n\big(Z^n\big) \geq 2\delta\big) \leq P_{Z^\infty}\big(M_{1,n}\big(Z^n\big) \geq \delta\big) + P_{Z^\infty}\big(M_{2,n}\big(Z^n\big) \geq \delta\big).$$

By following the proof of Lemma C.3(ii) of Chen and Pouzo (2012a), we have that $P_{Z^\infty}(M_{2,n}(Z^n) \geq \delta) < \delta/2$ eventually. Thus, to establish Result (1), it suffices to bound

$$P_{Z^\infty}\big(\big\{M_{1,n}\big(Z^n\big) \geq \delta\big\} \cap \big\{\lambda_{\min}\big(\big(P'P\big)/n\big) > c\big\}\big) + P_{Z^\infty}\big(\lambda_{\min}\big(\big(P'P\big)/n\big) \leq c\big).$$

By Assumption A.4(ii)(iii) and Theorem 1 in Newey (1997), $\lambda_{\min}((P'P)/n) \geq c > 0$ with probability P_{Z^∞} approaching 1, hence $P_{Z^\infty}(\lambda_{\min}((P'P)/n) \leq c) < \delta/4$ eventually. To bound the term corresponding to $M_{1,n}$, we note that[2]

$$\begin{aligned}
&\sum_{i=1}^n \big\| \widehat{m}^B(X_i, \alpha) - \widehat{m}(X_i, \alpha) - \big\{\widehat{m}^B(X_i, \alpha_0) - \widehat{m}(X_i, \alpha_0)\big\}\big\|_e^2 \\
&\quad = \sum_{i=1}^n \Delta\zeta^B(\alpha)' P\big(P'P\big)^- p^{J_n}(X_i) p^{J_n}(X_i)'\big(P'P\big)^- P'\Delta\zeta^B(\alpha) \\
&\quad = \Delta\zeta^B(\alpha)' P\big(P'P\big)^- P'\Delta\zeta^B(\alpha) \\
&\quad \leq \frac{1}{\lambda_{\min}\big(\big(P'P\big)/n\big)} \big\{n^{-1}\Delta\zeta^B(\alpha)' PP'\Delta\zeta^B(\alpha)\big\},
\end{aligned}$$

where $\Delta\zeta^B(\alpha) = ((\omega_1 - 1)\Delta\rho(Z_1, \alpha), \ldots, (\omega_n - 1)\Delta\rho(Z_n, \alpha))'$ with $\Delta\rho(Z, \alpha) \equiv \rho(Z, \alpha) - \rho(Z, \alpha_0)$. It is thus sufficient to show that, for large enough n,

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sup_{\mathcal{N}_{osn}} \frac{\overline{\tau}_n}{n^2}\Delta\zeta^B(\alpha)' PP'\Delta\zeta^B(\alpha) \geq M \Big| Z^n\right) \geq \delta\right) < \delta, \tag{C.10}$$

which is established in Lemma C.5.

[2]To ease the notational burden in the proof, we assume $d_\rho = 1$; when $d_\rho > 1$, the same proof steps hold, component by component.

For *Result (2)*, recall that $\ell_n^B(x,\alpha) \equiv \widetilde{m}(x,\alpha) + \widehat{m}^B(x,\alpha_0)$. By similar calculations to those in Ai and Chen (2003, p. 1824), it follows that

$$\begin{aligned} & E_{P_{V^\infty}}\left[n^{-1} \sum_{i=1}^{n} \left\| \widehat{m}^B(X_i,\alpha_0) \right\|_e^2 \right] \\ & \quad = E_{P_{V^\infty}}\left[p^{J_n}(X_i)' \left(P'P\right)^{-} P' E_{P_{V^\infty|X^\infty}}\left[\rho^B(\alpha_0)\rho^B(\alpha_0)'|X^n\right] \right. \\ & \qquad \left. \times P\left(P'P\right)^{-} p^{J_n}(X_i) \right], \end{aligned}$$

where $\rho^B(\alpha) \equiv (\rho^B(V_1,\alpha),\dots,\rho^B(V_n,\alpha))'$ with $\rho^B(V_i,\alpha) \equiv \omega_i \rho(Z_i,\alpha)$. Note that

$$\begin{aligned} & E_{P_{V|X^\infty}}\left[\rho^B(V_i,\alpha_0)\rho^B(V_j,\alpha_0)'|X^n\right] \\ & \quad = E_{P_\Omega}\left[\omega_i\omega_j E_{P_{V|X}}\left[\rho(Z_i,\alpha_0)\rho(Z_j,\alpha_0)'|X_i,X_j\right]\right] \\ & \quad = 0 \quad \text{for all } i \neq j, \end{aligned}$$

and

$$E_{P_{V|X^\infty}}\left[\rho^B(V_i,\alpha_0)\rho^B(V_i,\alpha_0)'|X^n\right] = \sigma_\omega^2 \Sigma_0(X_i).$$

So under Assumption Boot.1 or Boot.2, Assumptions 3.1(iv) and A.4(ii), applying the Markov inequality, we obtain: for all $\delta > 0$, there is an $M(\delta) > 0$ such that, for all $M \geq M(\delta)$,

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\frac{J_n}{n} n^{-1} \sum_{i=1}^{n} \left\| \widehat{m}^B(X_i,\alpha_0) \right\|_e^2 \geq M \,\middle|\, Z^n \right) \geq \delta \right) < \delta.$$

To establish Result (2), with $(\tau_n')^{-1} = \max\{\frac{J_n}{n}, b_{m,J_n}^2, (M_n\delta_n)^2\}$, it remains to show that

$$P_{Z^\infty}\left(\sup_{\mathcal{N}_{osn}} \tau_n' n^{-1} \sum_{i=1}^{n} \left\| \widetilde{m}(X_i,\alpha) \right\|_e^2 \geq M \right) < \delta. \tag{C.11}$$

By Lemma SM.1 of Chen and Pouzo (2012b), under Assumptions A.4 and A.5(i), we have: there are finite constants $c, c' > 0$ such that, for all $\delta > 0$, there is an $N(\delta)$ such that, for all $n \geq N(\delta)$,

$$\begin{aligned} & P_{Z^\infty}\Bigg(\forall \alpha \in \mathcal{N}_{osn} : c E_{P_X}\left[\left\| \widetilde{m}(X,\alpha) \right\|_e^2 \right] \\ & \quad \leq \frac{1}{n} \sum_{i=1}^{n} \left\| \widetilde{m}(X_i,\alpha) \right\|_e^2 \leq c' E_{P_X}\left[\left\| \widetilde{m}(X,\alpha) \right\|_e^2 \right] \Bigg) > 1 - \delta. \end{aligned}$$

Thus to show (C.11), it suffices to show that

$$\sup_{\mathcal{N}_{osn}} \tau_n' E_{P_X}\big[\big\|\widetilde{m}(X,\alpha)\big\|_e^2\big] = O(1).$$

By Assumption A.4(ii), it follows that

$$\begin{aligned}
&\sup_{\alpha\in\mathcal{N}_{osn}} E_{P_X}\big[\big\|\widetilde{m}(X,\alpha)\big\|_e^2\big] \\
&\quad \le \sup_{\mathcal{N}_{osn}} \big\{E_{P_X}\big[\big\|\widetilde{m}(X,\alpha)-m(X,\alpha)\big\|_e^2\big] + E_{P_X}\big[\big\|m(X,\alpha)\big\|_e^2\big]\big\} \\
&\quad \le \text{const.} \sup_{\alpha\in\mathcal{N}_{osn}} \max\big\{b_{m,J_n}^2, \|\alpha-\alpha_0\|^2\big\} = O\big(\big(\tau_n'\big)^{-1}\big),
\end{aligned}$$

where the last inequality follows from Assumptions A.4(ii)(iii)(iv) and 3.4. We thus obtain Result (2).

For *Result (3)*, we note that

$$\frac{1}{n}\sum_{i=1}^n \big\|\widehat{m}^B(X_i,\alpha)\big\|_{\widehat{\Sigma}^{-1}}^2 - \frac{1}{n}\sum_{i=1}^n \big\|\ell_n^B(X_i,\alpha)\big\|_{\widehat{\Sigma}^{-1}}^2 = R_{1n}^B(\alpha) + 2R_{2n}^B(\alpha),$$

where

$$\begin{aligned}
R_{1n}^B(\alpha) &\equiv \frac{1}{n}\sum_{i=1}^n \big\|\widehat{m}^B(X_i,\alpha) - \ell_n^B(X_i,\alpha)\big\|_{\widehat{\Sigma}^{-1}}^2, \\
R_{2n}^B(\alpha) &\le \sqrt{R_{1n}^B}\sqrt{\frac{1}{n}\sum_{i=1}^n \big\|\ell_n^B(X_i,\alpha)\big\|_{\widehat{\Sigma}^{-1}}^2}.
\end{aligned}$$

By Result (1) and Assumption 4.1(iii), we have

$$P_{Z^\infty}\Big(P_{V^\infty|Z^\infty}\Big(\sup_{\mathcal{N}_{osn}} \overline{\tau}_n R_{1n}^B(\alpha) \ge M \big| Z^n\Big) \ge \delta\Big) < \delta$$

with $\overline{\tau}_n^{-1} = \delta_n^2 (M_n \delta_{s,n})^{2\kappa} C_n$. By Results (1) and (2), and Assumption 4.1(iii), we have

$$P_{Z^\infty}\Big(P_{V^\infty|Z^\infty}\Big(\sup_{\mathcal{N}_{osn}} \widetilde{\tau}_n R_{2n}^B(\alpha) \ge M \big| Z^n\Big) \ge \delta\Big) < \delta$$

with $\widetilde{\tau}_n^{-1} \equiv M_n \delta_n^2 (M_n \delta_{s,n})^{\kappa} \sqrt{C_n}$. By Assumption A.5(iii) and the fact that L_n diverges, we obtain the desired result. *Q.E.D.*

In the following, we state Lemma C.5 and its proof.

LEMMA C.5: *Let Assumptions* 3.4(i)(ii), A.4(iii), A.5(i)(ii), *and either* Boot.1 *or* Boot.2 *hold*. *Then*: *for all* $\delta > 0$, *there is an* $M(\delta) > 0$ *such that*, *for all* $M \geq M(\delta)$,

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sup_{\mathcal{N}_{osn}} \frac{\overline{\tau}_n}{n^2} \Delta\zeta^B(\alpha)' PP' \Delta\zeta^B(\alpha) \geq M \middle| Z^n\right) \geq \delta\right) < 0.5\delta$$

eventually, *with* $\overline{\tau}_n^{-1} \equiv (\delta_n)^2 (M_n \delta_{s,n})^{2\kappa} C_n$, *where* $\Delta\zeta^B(\alpha) = ((\omega_1 - 1)\Delta\rho(Z_1, \alpha), \dots, (\omega_n - 1)\Delta\rho(Z_n, \alpha))'$ *and* $\Delta\rho(Z, \alpha) \equiv \rho(Z, \alpha) - \rho(Z, \alpha_0)$.

PROOF: Denote

$$M'_{1n}(Z^n) \equiv P_{V^\infty|Z^\infty}\left(\sup_{\mathcal{N}_{osn}} \frac{\overline{\tau}_n}{n^2} \Delta\zeta^B(\alpha)' PP' \Delta\zeta^B(\alpha) \geq M \middle| Z^n\right).$$

By the Markov inequality,

$$M'_{1n}(Z^n) \leq M^{-1} E_{P_{V^\infty|Z^\infty}}\left[\sup_{\mathcal{N}_{osn}} \frac{\overline{\tau}_n}{n^2} \Delta\zeta^B(\alpha)' PP' \Delta\zeta^B(\alpha)\right].$$

Hence it is sufficient to bound

$$\begin{aligned}
&P_{Z^\infty}\left(M'_{1n}(Z^n) \geq \delta\right) \\
&\quad \leq \frac{1}{M\delta} E_{P_{V^\infty}}\left[\sup_{\mathcal{N}_{osn}} \frac{\overline{\tau}_n}{n^2} \Delta\zeta^B(\alpha)' PP' \Delta\zeta^B(\alpha)\right] \\
&\quad = \frac{\overline{\tau}_n}{nM\delta} \sum_{j=1}^{J_n} E_{P_{V^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n (\omega_i - 1) f_j(Z_i, \alpha)\right)^2\right],
\end{aligned}$$

where the first inequality follows from the law of iterated expectations and the Markov inequality, and the second equality is due to the notation $f_j(z, \alpha) \equiv p_j(x)\{\rho(z, \alpha) - \rho(z, \alpha_0)\}$.

Under Assumption Boot.1, $\{(\omega_i - 1) f_j(Z_i, \alpha)\}_{i=1}^n$ are independent, and thus, by Proposition A.1.6 in Van der Vaart and Wellner (1996) (VdV-W),

$$\begin{aligned}
&\frac{\overline{\tau}_n}{nM\delta} \sum_{j=1}^{J_n} E_{P_{V^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left(n^{-1/2} \sum_{i=1}^n (\omega_i - 1) f_j(Z_i, \alpha)\right)^2\right] \\
&\quad \leq \frac{\overline{\tau}_n}{nM\delta} \sum_{j=1}^{J_n} \left(E_{P_{V^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left| n^{-1/2} \sum_{i=1}^n (\omega_i - 1) f_j(Z_i, \alpha)\right|\right]\right. \\
&\qquad \left. + \sqrt{E\left[\max_{i \leq n} \sup_{\mathcal{N}_{osn}} \left| n^{-1/2} (\omega_i - 1) f_j(Z_i, \alpha)\right|^2\right]}\right)^2.
\end{aligned}$$

The second term in the RHS is bounded above by

$$\sqrt{nn^{-1}E_{P_{V^\infty}}\Big[(\omega_i-1)^2\sup_{\mathcal{N}_{osn}}\big|f_j(Z_i,\alpha)\big|^2\Big]}$$
$$\leq\sqrt{E_{P_\omega}\big[(\omega_i-1)^2\big]E_{P_{Z^\infty}}\Big[\sup_{\mathcal{N}_{osn}}\big|f_j(Z_i,\alpha)\big|^2\Big]}=O\big((M_n\delta_{s,n})^\kappa\big)$$

by Assumptions A.4(iii), A.5(ii), and Boot.1. Hence, under Assumption Boot.1, we need to control

$$\frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n}\left(E_{P_{V^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left|n^{-1/2}\sum_{i=1}^{n}(\omega_i-1)f_j(Z_i,\alpha)\right|\right]\right)^2 + O\left(\frac{\overline{\tau}_n J_n}{nM\delta}(M_n\delta_{s,n})^{2\kappa}\right). \tag{C.12}$$

Under Assumption Boot.2, $((\omega_i-1)f_j(Z_i,\alpha))_i$ are *not* independent. So we need to take some additional steps to arrive to an equation of the form of (C.12). Under Assumption Boot.2, it follows that

$$\frac{\overline{\tau}_n}{M\delta}\sum_{j=1}^{J_n}E_{P_{V^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left(n^{-1}\sum_{i=1}^{n}(\omega_i-1)f_j(Z_i,\alpha)\right)^2\right]$$
$$=\frac{\overline{\tau}_n}{M\delta}\sum_{j=1}^{J_n}E_{P_{V^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left(n^{-1}\sum_{i=1}^{n}\omega_i f_j(Z_i,\alpha)-n^{-1}\sum_{i=1}^{n}f_j(Z_i,\alpha)\right)^2\right]$$
$$=\frac{\overline{\tau}_n}{M\delta}\sum_{j=1}^{J_n}E_{P_{Z^\infty}\times P_{\widehat{Z}^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left(n^{-1}\sum_{i=1}^{n}(\delta_{\widehat{Z}_i}-\mathbb{P}_n)\big[f_j(\cdot,\alpha)\big]\right)^2\right],$$

where the last line follows from the fact that ω_i are the number of times the variable Z_i appears on the bootstrap sample. Thus, the distribution of $\omega_i\delta_{Z_i}$ is the same as that of $\delta_{\widehat{Z}_i}$ where $(\widehat{Z}_i)_i$ is the bootstrap sample, that is, an i.i.d. sample from $\mathbb{P}_n\equiv n^{-1}\sum_{i=1}^{n}\delta_{Z_i}$. By a slight adaptation of Lemma 3.6.6 in VdV-W (allowing for square of the norm), it follows that

$$E_{P_{Z^\infty}\times P_{\widehat{Z}^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left(n^{-1}\sum_{i=1}^{n}(\delta_{\widehat{Z}_i}-\mathbb{P}_n)\big[f_j(\cdot,\alpha)\big]\right)^2\right]$$
$$\leq E_{P_{Z^\infty}}\left[E_{P_{\widetilde{N}^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left(n^{-1}\sum_{i=1}^{n}\widetilde{N}_i\delta_{Z_i}\big[f_j(\cdot,\alpha)\big]\right)^2\right]\right],$$

where $\widetilde{N}_i = N_i - N_i'$ with N_i and N_i' being i.i.d. Poisson variables with parameter 0.5 ($P_{\widetilde{N}^\infty}$ is the corresponding probability). Note that now, $\{\widetilde{N}_i f_j(Z_i,\alpha)\}_{i=1}^n$ are independent. So by Proposition A.1.6 in VdV-W,

$$\begin{aligned}
&\frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n} E_Q\left[\sup_{\mathcal{N}_{osn}}\left(n^{-1/2}\sum_{i=1}^n \widetilde{N}_i f_j(Z_i,\alpha)\right)^2\right] \\
&\quad\le \frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n}\left(E_Q\left[\sup_{\mathcal{N}_{osn}}\left|n^{-1/2}\sum_{i=1}^n \widetilde{N}_i f_j(Z_i,\alpha)\right|\right]\right. \\
&\qquad \left. + \sqrt{E\left[\max_{i\le n}\sup_{\mathcal{N}_{osn}}\left|n^{-1/2}\widetilde{N}_i f_j(Z_i,\alpha)\right|^2\right]}\right)^2,
\end{aligned}$$

where $Q \equiv P_{Z^\infty}\times P_{\widetilde{N}^\infty}$. By the Cauchy–Schwarz inequality, the second term in the RHS is bounded above by

$$\begin{aligned}
&\sqrt{nn^{-1}E_Q\left[|\widetilde{N}|^2\sup_{\mathcal{N}_{osn}}\left|f_j(Z,\alpha)\right|^2\right]} \\
&\quad\le \sqrt{E_{P_{\widetilde{N}}}\left[|\widetilde{N}|^2\right]E_{P_Z}\left[\sup_{\mathcal{N}_{osn}}\left|f_j(Z,\alpha)\right|^2\right]} = O\big((M_n\delta_{s,n})^\kappa\big)
\end{aligned}$$

by Assumptions A.4(iii) and A.5(ii) and $E[|\widetilde{N}|^2]<\infty$. Therefore, under Assumption Boot.2, we need to control

$$\frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n}\left(E_Q\left[\sup_{\mathcal{N}_{osn}}\left|n^{-1/2}\sum_{i=1}^n \widetilde{N}_i f_j(Z_i,\alpha)\right|\right]\right)^2 + O\left(\frac{\overline{\tau}_n J_n}{nM\delta}(M_n\delta_{s,n})^{2\kappa}\right). \tag{C.13}$$

Applying Lemma 2.9.1 of VdV-W, we can bound the leading terms in equations (C.12) and (C.13) respectively as follows:

$$\begin{aligned}
&\frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n} E_{P_{V^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left|n^{-1/2}\sum_{i=1}^n(\omega_i-1)\delta_{Z_i}\big[f_j(\cdot,\alpha)\big]\right|\right] \\
&\quad\le \frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n}\left\{\int_0^\infty\sqrt{P\big(|\omega-1|\ge t\big)}\,dt\right\} \\
&\qquad\times\max_{1\le l\le n}E_{P_{Z^\infty}\times P_{\epsilon^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left|l^{-1/2}\sum_{i=1}^l\epsilon_i\delta_{Z_i}\big[f_j(\cdot,\alpha)\big]\right|\right],
\end{aligned} \tag{C.14}$$

and

$$
\text{(C.15)} \quad \frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n} E_{P_{Z^\infty}}\left[E_{P_{\widetilde{N}}}\left[\sup_{\mathcal{N}_{osn}}\left|n^{-1/2}\sum_{i=1}^{n}\widetilde{N}_i\delta_{Z_i}[f_j(\cdot,\alpha)]\right|\right]\right]
$$

$$
\le \frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n}\left\{\int_0^\infty \sqrt{P(|\widetilde{N}|\ge t)}\,dt\right\}
$$

$$
\times \max_{1\le l\le n} E_{P_{Z^\infty}\times P_{\epsilon^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left|l^{-1/2}\sum_{i=1}^{l}\epsilon_i\delta_{Z_i}[f_j(\cdot,\alpha)]\right|\right],
$$

where $(\epsilon_i)_{i=1}^n$ is a sequence of Rademacher random variables. Note that $\{\int_0^\infty\sqrt{P(|\omega-1|\ge t)}\,dt\} < \infty$ (under Assumption Boot.1), and also $\{\int_0^\infty\sqrt{P(|\widetilde{N}|\ge t)}\,dt\} \le 2\sqrt{2}$ (see VdV-W, p. 351). Hence in both cases we need to bound

$$
\text{(C.16)} \quad \frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n}\left(\max_{1\le l\le n} E_{P_{Z^\infty}\times P_{\epsilon^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left|l^{-1/2}\sum_{i=1}^{l}\epsilon_i\delta_{Z_i}[f_j(\cdot,\alpha)]\right|\right]\right)^2
$$

$$
\le \frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n}\left(\max_{1\le l\le n} E_{P_{Z^\infty}\times P_{\epsilon^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left|l^{-1/2}\sum_{i=1}^{l}\epsilon_i\delta_{Z_i}[\overline{f}_j(\cdot,\alpha)]\right|\right]\right.
$$

$$
\left.+\max_{1\le l\le n} E_{P_{Z^\infty}\times P_{\epsilon^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left|l^{-1/2}\sum_{i=1}^{l}\epsilon_i E_{P_Z}[f_j(Z,\alpha)]\right|\right]\right)^2
$$

$$
\le 2T_{1,n} + 2T_{2,n},
$$

where $\overline{f}_j(Z,\alpha) = f_j(Z,\alpha) - E_{P_Z}[f_j(Z,\alpha)]$,

$$
T_{1,n} = \frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n}\left(\max_{1\le l\le n} E_{P_{Z^\infty}\times P_{\epsilon^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left|l^{-1/2}\sum_{i=1}^{l}\epsilon_i\delta_{Z_i}[\overline{f}_j(\cdot,\alpha)]\right|\right]\right)^2
$$

and

$$
T_{2,n} = \frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n}\left(\max_{1\le l\le n} E_{P_{Z^\infty}\times P_{\epsilon^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left|l^{-1/2}\sum_{i=1}^{l}\epsilon_i E_{P_Z}[f_j(Z,\alpha)]\right|\right]\right)^2.
$$

To bound the term $T_{2,n}$, we note that

$$\begin{aligned}
&\max_{1\le j\le J_n}\max_{1\le l\le n} E_{P_{Z^\infty}\times P_{\epsilon^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left|l^{-1/2}\sum_{i=1}^{l}\epsilon_i E_{P_Z}\big[f_j(Z,\alpha)\big]\right|\right]\\
&\quad=\max_{1\le j\le J_n}\max_{1\le l\le n}\sup_{\mathcal{N}_{osn}}\big|E_{P_Z}\big[f_j(Z,\alpha)\big]\big|E_{P_{\epsilon^\infty}}\left[\left|l^{-1/2}\sum_{i=1}^{l}\epsilon_i\right|\right]\\
&\quad\le\max_{1\le j\le J_n}\max_{1\le l\le n}\sup_{\mathcal{N}_{osn}}\big|E_{P_X}\big[p_j(X)\Delta m(X,\alpha)\big]\big|\sqrt{E_{P_{\epsilon^\infty}}\left[\left(l^{-1/2}\sum_{i=1}^{l}\epsilon_i\right)^2\right]}\\
&\quad\le\max_{1\le j\le J_n}\max_{1\le l\le n}\Bigg(\sqrt{E_{P_Z}\big[\big|p_j(X)\big|^2\big]}\sup_{\mathcal{N}_{osn}}\sqrt{E_{P_X}\big[\big|\Delta m(X,\alpha)\big|^2\big]}\\
&\qquad\times\sqrt{E_{P_{\epsilon^\infty}}\left[l^{-1}\sum_{i=1}^{l}(\epsilon_i)^2\right]}\Bigg)\\
&\quad=O(M_n\delta_n),
\end{aligned}$$

where $\Delta m(X,\alpha)\equiv m(X,\alpha)-m(X,\alpha_0)$ and the inequality follows from Cauchy–Schwarz and the fact that ϵ_i are independent, and the last two equal signs are due to Assumptions 3.4(i)(ii) and A.4(iii). Thus $T_{2,n}\le$ const. $\times$ $(M_n\delta_n)^2\frac{\overline{\tau}_n J_n}{nM\delta}$.

To bound the term $T_{1,n}$, we note that by the "desymmetrization lemma" 2.3.6 in VdV-W (note that $\overline{f}_j(Z_i,\alpha)$ are centered),

$$T_{1,n}\le\text{const.}\times\frac{\overline{\tau}_n}{nM\delta}\sum_{j=1}^{J_n}\max_{1\le l\le n}\left(E_{P_{Z^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left|l^{-1/2}\sum_{i=1}^{l}\overline{f}_j(Z_i,\alpha)\right|\right]\right)^2.$$

By Van der Vaart and Wellner (1996, Theorem 2.14.2), we have (up to some omitted constant), for all j,

$$\begin{aligned}
&E_{P_{Z^\infty}}\left[\sup_{\alpha\in\mathcal{N}_{osn}}\left|l^{-1/2}\sum_{i=1}^{l}\overline{f}_j(Z_i,\alpha)\right|\right]\\
&\quad\le\left\{(M_n\delta_{s,n})^\kappa\int_0^1\sqrt{1+\log N_{[]}\big(w(M_n\delta_{s,n})^\kappa,\mathcal{E}_{ojn},\|\cdot\|_{L^2(f_Z)}\big)}\,dw\right\},
\end{aligned}$$

where $\mathcal{E}_{ojn}=\{p_j(\cdot)(\rho(\cdot,\alpha)-\rho(\cdot,\alpha_0))-E[p_j(\cdot)(\rho(\cdot,\alpha)-\rho(\cdot,\alpha_0))]:\alpha\in\mathcal{N}_{osn}\}$.

Given any $w > 0$, let $(\{g_l^m, g_u^m\})_{m=1,\dots,N(w)}$ be the $\|\cdot\|_{L^2(f_Z)}$-norm brackets of $\mathcal{O}_{on}$. If $\{\rho(\cdot,\alpha) - \rho(\cdot,\alpha_0)\} \in \mathcal{O}_{on}$ belongs to a bracket $\{g_l^m, g_u^m\}$, then, since $|p_j(x)| < \text{const.} < \infty$ by Assumption A.4(iii),

$$g_l^m(Z) \le p_j(X)\{\Delta\rho(Z,\alpha)\} \le g_u^m(Z)$$

(where $\{g_l^m, g_u^m\}$ are transformations of the original ones, given by $(1\{p_j > 0\}g_l^m + 1\{p_j \le 0\}g_u^m)p_j$ and $(1\{p_j > 0\}g_u^m + 1\{p_j \le 0\}g_l^m)p_j$ and since $|p_j(x)| < \text{const.} < \infty$ the $\|\cdot\|_{L^2(f_Z)}$-norm of the new brackets is given by $\delta \times 2\text{const.}$. We keep the same notation and omit the constant "2const." to ease the notational burden), and from the previous calculations it is easy to see that

$$\begin{aligned}\{g_l^m(Z) - E[g_u^m(Z)]\} &\le p_j(X)\Delta\rho(Z,\alpha) - E[p_j(X)\Delta\rho(Z,\alpha)] \\ &\le \{g_u^m(Z) - E[g_l^m(Z)]\}.\end{aligned}$$

So functions of the form

$$\big(\{(g_l^m(Z) - E[g_u^m(Z)]), (g_u^m(Z) - E[g_l^m(Z)])\}\big)_{m=1,\dots,N(w)}$$

form $\|\cdot\|_{L^2(f_V)}$-norm brackets on $\mathcal{E}_{ojn}$. By construction, $N_{[]}(w, \mathcal{E}_{ojn}, \|\cdot\|_{L^2(f_Z)}) \le N(w)$. Hence (up to some omitted constants)

$$\begin{aligned}&E_{P_{Z^\infty}}\left[\sup_{\alpha\in\mathcal{N}_{osn}}\left|l^{-1/2}\sum_{i=1}^{l}\overline{f}_j(Z_i,\alpha)\right|\right] \\ &\quad\le (M_n\delta_{s,n})^\kappa \\ &\qquad\times \max_{j=1,\dots,J_n}\left\{\int_0^1\sqrt{1+\log N_{[]}\big(w(M_n\delta_{s,n})^\kappa, \mathcal{O}_{on}, \|\cdot\|_{L^2(f_Z)}\big)}\,dw\right\} \\ &\quad\le (M_n\delta_{s,n})^\kappa\sqrt{C_n},\end{aligned}$$

where the last inequality follows from Assumption A.5(ii). Notice that the above RHS does not depend on l nor on j, so we obtain

$$\text{(C.17)}\qquad \max_{1\le j\le J_n}\max_{1\le l\le n} E_{P_{Z^\infty}}\left[\sup_{\alpha\in\mathcal{N}_{osn}}\left(l^{-1/2}\sum_{i=1}^{l}\overline{f}_j(Z_i,\alpha)\right)^2\right] \le \text{const.}\times(M_n\delta_{s,n})^{2\kappa}C_n$$

and hence $T_{1,n} \le \text{const.} \times (M_n\delta_{s,n})^{2\kappa}C_n\frac{\overline{\tau}_n J_n}{nM\delta}$.

Note that $\max\{(M_n\delta_n)^2, (M_n\delta_{s,n})^{2\kappa}\} = (M_n\delta_{s,n})^{2\kappa}$ (by assumption) and that $\overline{\tau}_n^{-1} \equiv \frac{J_n}{n}(M_n\delta_{s,n})^{2\kappa}C_n$; the desired result follows. *Q.E.D.*

PROOF OF LEMMA A.3: Denote

$$T_{n\mathrm{I}}^B \equiv \sup_{\mathcal{N}_{osn}} \left| \frac{1}{n} \sum_{i=1}^n \left(\frac{d\widetilde{m}(X_i, \alpha)}{d\alpha}[u_n^*] \right)' \widehat{\Sigma}(X_i)^{-1} \ell_n^B(X_i, \alpha) - \frac{1}{n} \sum_{i=1}^n \left(\frac{dm(X_i, \alpha_0)}{d\alpha}[u_n^*] \right)' \Sigma(X_i)^{-1} \ell_n^B(X_i, \alpha) \right|,$$

and

$$T_{n\mathrm{II}}^B \equiv \sup_{\mathcal{N}_{osn}} \left| \frac{1}{n} \sum_{i=1}^n \left(\frac{dm(X_i, \alpha_0)}{d\alpha}[u_n^*] \right)' \Sigma(X_i)^{-1} \ell_n^B(X_i, \alpha) - \left\{ \mathbb{Z}_n^\omega + \langle u_n^*, \alpha - \alpha_0 \rangle \right\} \right|.$$

It suffices to show that for all $\delta > 0$, there is $N(\delta)$ such that, for all $n \geq N(\delta)$,

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sqrt{n}T_{n\mathrm{I}}^B \geq \delta | Z^n\right) \geq \delta\right) < \delta \tag{C.18}$$

and

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sqrt{n}T_{n\mathrm{II}}^B \geq \delta | Z^n\right) \geq \delta\right) < \delta. \tag{C.19}$$

We first verify equation (C.18). Note that

$$\begin{aligned} T_{n\mathrm{I}}^B &\leq \sup_{\mathcal{N}_{osn}} \left| \frac{1}{n} \sum_{i=1}^n \left(\frac{d\widetilde{m}(X_i, \alpha)}{d\alpha}[u_n^*] - \frac{dm(X_i, \alpha_0)}{d\alpha}[u_n^*] \right)' \times \widehat{\Sigma}(X_i)^{-1} \ell_n^B(X_i, \alpha) \right| \\ &\quad + \sup_{\mathcal{N}_{osn}} \left| \frac{1}{n} \sum_{i=1}^n \left(\frac{dm(X_i, \alpha_0)}{d\alpha}[u_n^*] \right)' \times \left\{ \widehat{\Sigma}(X_i)^{-1} - \Sigma(X_i)^{-1} \right\} \ell_n^B(X_i, \alpha) \right| \\ &\equiv T_{n\mathrm{Ia}}^B + T_{n\mathrm{Ib}}^B. \end{aligned}$$

By Assumption 4.1(iii) and the Cauchy–Schwarz inequality, it follows that, for some $C \in (0, \infty)$,

$$\begin{aligned}
&P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sqrt{n}T^B_{n\mathrm{Ia}} \geq \delta|Z^n\right) \geq \delta\right) \\
&\leq P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sup_{\mathcal{N}_{osn}} \sqrt{\frac{\sum_{i=1}^n \left\| \frac{dm(X_i,\alpha_0)}{d\alpha}[u^*_n] - \frac{d\widetilde{m}(X_i,\alpha)}{d\alpha}[u^*_n]\right\|^2_e}{n}} \right.\right. \\
&\qquad \left.\left. \times \sqrt{\frac{\sum_{i=1}^n \left\|\ell^B_n(X_i,\alpha)\right\|^2_e}{n}} \geq \frac{C\delta}{\sqrt{n}} \middle| Z^n\right) \geq \delta\right) \\
&\qquad + P_{Z^\infty}\left(\lambda_{\min}\left(\widehat{\Sigma}(X)\right) < c\right).
\end{aligned}$$

The second term in the RHS vanishes eventually, so we focus on the first term. It follows that

$$\begin{aligned}
&P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sup_{\mathcal{N}_{osn}} \sqrt{\frac{\sum_{i=1}^n \left\| \frac{dm(X_i,\alpha_0)}{d\alpha}[u^*_n] - \frac{d\widetilde{m}(X_i,\alpha)}{d\alpha}[u^*_n]\right\|^2_e}{n}} \right.\right. \\
&\qquad \left.\left. \times \sqrt{\frac{1}{n}\sum_{i=1}^n \left\|\ell^B_n(X_i,\alpha)\right\|^2_e} \geq \frac{C\delta}{\sqrt{n}} \middle| Z^n\right) \geq \delta\right) \\
&\leq P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sup_{\mathcal{N}_{osn}} \sqrt{\frac{1}{n}\sum_{i=1}^n \left\| \frac{dm(X_i,\alpha_0)}{d\alpha}[u^*_n] - \frac{d\widetilde{m}(X_i,\alpha)}{d\alpha}[u^*_n]\right\|^2_e}\right.\right. \\
&\qquad \left.\left. \times \sqrt{\frac{Mn}{\tau'_n}} \geq C\delta \middle| Z^n\right) \geq 0.5\delta\right)
\end{aligned}$$

$$+ P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sup_{\mathcal{N}_{osn}}\sqrt{\frac{\tau_n'}{n}\sum_{i=1}^n \left\|\ell_n^B(X_i,\alpha)\right\|_e^2} \geq \sqrt{M}\middle|Z^n\right) \geq 0.5\delta\right).$$

By Lemma A.2(2), the second term on the RHS is less than 0.5δ eventually (with $(\tau_n')^{-1} = \text{const.}\,(M_n\delta_n)^2$). Regarding the first term, note that

$$\begin{aligned}
&\sup_{\mathcal{N}_{osn}}\sqrt{\frac{1}{n}\sum_{i=1}^n\left\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*] - \frac{d\widetilde{m}(X_i,\alpha)}{d\alpha}[u_n^*]\right\|_e^2}\sqrt{\frac{n}{\tau_n'}} \\
&\quad\leq \sup_{\mathcal{N}_{osn}}\sqrt{\frac{1}{n}\sum_{i=1}^n\left\|\frac{d\widetilde{m}(X_i,\alpha_0)}{d\alpha}[u_n^*] - \frac{d\widetilde{m}(X_i,\alpha)}{d\alpha}[u_n^*]\right\|_e^2 \times \frac{n}{\tau_n'}} \\
&\qquad+ \sup_{\mathcal{N}_{osn}}\sqrt{\frac{1}{n}\sum_{i=1}^n\left\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*] - \frac{d\widetilde{m}(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|_e^2 \times \frac{n}{\tau_n'}} \\
&\quad\leq \sup_{\mathcal{N}_{osn}}\sqrt{\frac{1}{n}\sum_{i=1}^n\left\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*] - \frac{dm(X_i,\alpha)}{d\alpha}[u_n^*]\right\|_e^2 \times \frac{n}{\tau_n'}} \\
&\qquad+ o_{P_{Z^\infty}}(1),
\end{aligned}$$

by the LS projection property and the definition of $\widetilde{m}$, as well as by the Markov inequality and Assumption A.6(i). Next, by the Markov inequality and Assumption A.7(ii), we have

$$\begin{aligned}
&P_{Z^\infty}\left(\sup_{\mathcal{N}_{osn}}\sqrt{\frac{1}{n}\sum_{i=1}^n\left\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*] - \frac{dm(X_i,\alpha)}{d\alpha}[u_n^*]\right\|_e^2}\sqrt{\frac{n}{\tau_n'}} \geq 0.5\delta\right) \\
&\quad\leq \frac{2}{\delta}\sqrt{E_{P_{Z^\infty}}\left[\sup_{\mathcal{N}_{osn}}\left\|\frac{dm(X,\alpha_0)}{d\alpha}[u_n^*] - \frac{dm(X,\alpha)}{d\alpha}[u_n^*]\right\|_e^2\right] \times \frac{n}{\tau_n'}} \to 0.
\end{aligned}$$

Thus, we established that

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sqrt{n}T_{n\mathrm{Ia}}^B \geq \delta|Z^n\right) \geq \delta\right) < \delta \quad \text{eventually.}$$

By similar arguments, Assumptions 4.1(iii) and A.5(iv), Lemma A.2(2), and that $\frac{1}{n}\sum_{i=1}^n \|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\|_e^2$ is bounded in probability, it can be shown that

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sqrt{n}T_{n\mathrm{Ib}}^B \geq \delta|Z^n\right) \geq \delta\right) < \delta, \quad \text{eventually.}$$

Therefore, we establish equation (C.18).

For equation (C.19), let $g(X, u_n^*) \equiv (\frac{dm(X,\alpha_0)}{d\alpha}[u_n^*])'\Sigma^{-1}(X)$. Then

$$\begin{aligned} T_{n\mathrm{II}}^B &\leq \sup_{\mathcal{N}_{osn}} \left| \frac{1}{n} \sum_{i=1}^n g(X_i, u_n^*) \widetilde{m}(X_i, \alpha) - \langle u_n^*, \alpha - \alpha_0 \rangle \right| \\ &\quad + \left| \frac{1}{n} \sum_{i=1}^n g(X_i, u_n^*) \widehat{m}^B(X_i, \alpha_0) - \mathbb{Z}_n^\omega \right| \\ &\equiv T_{n\mathrm{IIa}} + T_{n\mathrm{IIb}}^B. \end{aligned}$$

Thus, to show equation (C.19), it suffices to show that $\sqrt{n} T_{n\mathrm{IIa}} = o_{P_{Z^\infty}}(1)$ and that

$$P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sqrt{n} T_{n\mathrm{IIb}}^B \geq \delta | Z^n\right) \geq \delta\right) < \delta \quad \text{eventually.} \tag{C.20}$$

First we consider the term $T_{n\mathrm{IIa}}$. This part of the proof is similar to those in Ai and Chen (2003), Ai and Chen (2007), and Chen and Pouzo (2009) for their regular functional $\lambda'\theta$ case, and hence we shall be brief. By the orthogonality properties of the LS projection and the definition of $\widetilde{m}(X_i, \alpha)$ and $\widetilde{g}(X_i, u_n^*)$, we have

$$n^{-1} \sum_{i=1}^n g(X_i, u_n^*) \widetilde{m}(X_i, \alpha) = n^{-1} \sum_{i=1}^n \widetilde{g}(X_i, u_n^*) m(X_i, \alpha).$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned} &\sup_{\mathcal{N}_{osn}} \left| \frac{1}{n} \sum_{i=1}^n \{\widetilde{g}(X_i, u_n^*) - g(X_i, u_n^*)\}\{m(X_i, \alpha) - m(X_i, \alpha_0)\} \right| \\ &\quad \leq \sqrt{\frac{1}{n} \sum_{i=1}^n \left\| \widetilde{g}(X_i, u_n^*) - g(X_i, u_n^*) \right\|_e^2} \\ &\qquad \times \sup_{\mathcal{N}_{osn}} \sqrt{\frac{1}{n} \sum_{i=1}^n \left\| m(X_i, \alpha) - m(X_i, \alpha_0) \right\|_e^2}. \end{aligned}$$

By Assumption A.6(iii),

$$\begin{aligned} &\sqrt{n} \sup_{\mathcal{N}_{osn}} \frac{1}{n} \sum_{i=1}^n \left\{ \left\| m(X_i, \alpha) - m(X_i, \alpha_0) \right\|_e^2 \right. \\ &\quad \left. - E_{P_X}\left[\left\| m(X_1, \alpha) - m(X_1, \alpha_0) \right\|_e^2 \right] \right\} = o_P(1). \end{aligned}$$

Thus, since $\sup_{\mathcal{N}_{osn}} E_{P_X}[\|m(X_1,\alpha)-m(X_1,\alpha_0)\|_e^2] = O(M_n^2\delta_n^2)$, it follows that

$$\sup_{\mathcal{N}_{osn}} \frac{1}{n}\sum_{i=1}^{n}\|m(X_i,\alpha)-m(X_i,\alpha_0)\|_e^2 = O_{P_{Z^\infty}}\big((M_n\delta_n)^2 + o_{P_{Z^\infty}}(n^{-1/2})\big).$$

This, Assumption A.6(ii), and $\delta_n = o(n^{-1/4})$ (by Assumption A.5(iv)) imply that

$$\begin{aligned}&\sup_{\mathcal{N}_{osn}}\left|\frac{1}{n}\sum_{i=1}^{n}\{\widetilde{g}(X_i,u_n^*)-g(X_i,u_n^*)\}\{m(X_i,\alpha)-m(X_i,\alpha_0)\}\right|\\ &\quad\le o_{P_{Z^\infty}}\left(\frac{1}{\sqrt{n}M_n\delta_n}\right)\times O_{P_{Z^\infty}}\left(\sqrt{(M_n\delta_n)^2+o(n^{-1/2})}\right) = o_{P_{Z^\infty}}(n^{-1/2}).\end{aligned}$$

Therefore,

$$\begin{aligned}\sqrt{n}T_{n\mathrm{IIa}} &= \sqrt{n}\sup_{\mathcal{N}_{osn}}\left|\frac{1}{n}\sum_{i=1}^{n} g(X_i,u_n^*)m(X_i,\alpha) - \langle u_n^*,\alpha-\alpha_0\rangle\right|\\ &\quad + o_{P_{Z^\infty}}(n^{-1/2}).\end{aligned}$$

By Assumption A.6(iv), $\sqrt{n}\sup_{\mathcal{N}_{osn}}|\frac{1}{n}\sum_{i=1}^{n} g(X_i,u_n^*)m(X_i,\alpha) - E_{P_X}[g(X_1, u_n^*)\{m(X_1,\alpha)-m(X_1,\alpha_0)\}]| = o_{P_{Z^\infty}}(1)$. Thus, by Assumption A.7(iv), we conclude that $\sqrt{n}T_{n\mathrm{IIa}} = o_{P_{Z^\infty}}(1)$.

Next we consider the term $T_{n\mathrm{IIb}}^B$. By the orthogonality properties of the LS projection,

$$n^{-1}\sum_{i=1}^{n} g(X_i,u_n^*)\widehat{m}^B(X_i,\alpha_0) = n^{-1}\sum_{i=1}^{n}\widetilde{g}(X_i,u_n^*)\rho^B(V_i,\alpha_0),$$

where $\rho^B(V_i,\alpha_0)\equiv \omega_{i,n}\rho(Z_i,\alpha_0)$ and $\{\omega_{i,n}\}_{i=1}^n$ is independent of $\{Z_i\}_{i=1}^n$.

Hence, by applying the Markov inequality twice, it follows that

$$\begin{aligned}&P_{Z^\infty}\big(P_{V^\infty|Z^\infty}(\sqrt{n}T_{n\mathrm{IIb}}^B \ge \delta|Z^n)\ge\delta\big)\\ &\quad\le \delta^{-4}E_{P_{V^\infty}}\left[n^{-1}\left(\sum_{i=1}^{n}\{g(X_i,u_n^*)-\widetilde{g}(X_i,u_n^*)\}\rho^B(V_i,\alpha_0)\right)^2\right].\end{aligned}$$

Regarding the cross-products terms where $i\neq j$, note that

$$\begin{aligned}&E_{P_{V^\infty}}\big[\{g(X_j,u_n^*)-\widetilde{g}(X_j,u_n^*)\}\{g(X_i,u_n^*)-\widetilde{g}(X_i,u_n^*)\}\\ &\qquad\times\rho^B(V_i,\alpha_0)\rho^B(V_j,\alpha_0)\big]\end{aligned}$$

$$
\begin{aligned}
&= E_{P_{V^\infty}}\big[\big\{g\big(X_j, u_n^*\big) - \widetilde{g}\big(X_j, u_n^*\big)\big\}\big\{g\big(X_i, u_n^*\big) - \widetilde{g}\big(X_i, u_n^*\big)\big\} \\
&\quad \times E_{P_{V^\infty|X^\infty}}\big[\rho^B(V_i, \alpha_0)\rho^B(V_j, \alpha_0)|X^n\big]\big] \\
&= E_{P_{V^\infty}}\big[\big\{g\big(X_j, u_n^*\big) - \widetilde{g}\big(X_j, u_n^*\big)\big\}\big\{g\big(X_i, u_n^*\big) - \widetilde{g}\big(X_i, u_n^*\big)\big\} \\
&\quad \times E_{P_{V^\infty|X^\infty}}\big[\omega_i\omega_j|X^n\big]E_{P_{Z^\infty|X^\infty}}\big[\rho(Z_i, \alpha_0)\rho(Z_j, \alpha_0)|X^n\big]\big] \\
&= 0,
\end{aligned}
$$

since $E_{P_{Z^\infty|X^\infty}}[\rho(Z_i, \alpha_0)\rho(Z_j, \alpha_0)|X^n] = E_{P_{Z|X}}[\rho(Z_i, \alpha_0)|X_i]E_{P_{Z|X}}[\rho(Z_j, \alpha_0)|X_j] = 0$ for $i \neq j$. Thus, it suffices to study

$$
\begin{aligned}
&\delta^{-4}E_{P_{V^\infty}}\left[n^{-1}\sum_{i=1}^{n}\big(g\big(X_i, u_n^*\big) - \widetilde{g}\big(X_i, u_n^*\big)\big)^2\big(\rho^B(V_i, \alpha_0)\big)^2\right] \\
&= \delta^{-4}n^{-1}\sum_{i=1}^{n}E_{P_{V^\infty}}\big[\big(g\big(X_i, u_n^*\big) - \widetilde{g}\big(X_i, u_n^*\big)\big)^2 \\
&\quad \times E_{P_{V^\infty|X^\infty}}\big[\big(\omega_i\rho(Z_i, \alpha_0)\big)^2|X^n\big]\big].
\end{aligned}
$$

By the original-sample $\{Z_i\}_{i=1}^n$ being i.i.d., $\{\omega_{i,n}\}_{i=1}^n$ being independent of $\{Z_i\}_{i=1}^n$, Assumption 3.1(iv), and the fact that $\sigma_\omega^2 < \infty$, we can majorize the previous expression (up to an omitted constant) by

$$
\delta^{-4}E_{P_{V^\infty}}\big[\big(g\big(X_i, u_n^*\big) - \widetilde{g}\big(X_i, u_n^*\big)\big)^2\big] = o(1),
$$

where the last equality is due to Assumption A.6(ii). Hence we established equation (C.20). The desired result now follows. *Q.E.D.*

PROOF OF LEMMA A.4: By the Cauchy–Schwarz inequality and Assumption 4.1(iii), it suffices to show that

$$
\begin{aligned}
&P_{Z^\infty}\left(P_{V^\infty|Z^\infty}\left(\sup_{\mathcal{N}_{osn}}\sqrt{n^{-1}\sum_{i=1}^{n}\left\|\frac{d^2\widetilde{m}(X_i, \alpha)}{d\alpha^2}\big[u_n^*, u_n^*\big]\right\|_e^2}\right.\right. \\
&\quad \left.\left.\times \sup_{\mathcal{N}_{osn}}\sqrt{n^{-1}\sum_{i=1}^{n}\big\|\ell_n^B(X_i, \alpha)\big\|_e^2} \geq \delta\,\Big|\,Z^n\right) \geq \delta\right) < \delta.
\end{aligned}
$$

By Lemma A.2(2), it suffices to show that

$$
P_{Z^\infty}\left(\sup_{\mathcal{N}_{osn}}\sqrt{n^{-1}\sum_{i=1}^{n}\left\|\frac{d^2\widetilde{m}(X_i, \alpha)}{d\alpha^2}\big[u_n^*, u_n^*\big]\right\|_e^2} \geq \frac{\delta}{M_n\delta_n}\right) < \delta.
$$

By the Markov inequality and the LS projection properties, the LHS of the previous equation can be bounded above by

$$\frac{M_n^2\delta_n^2}{\delta^2}E_{P_X}\left[\sup_{\mathcal{N}_{osn}}\left\|\frac{d^2\widetilde{m}(X,\alpha)}{d\alpha^2}[u_n^*,u_n^*]\right\|_e^2\right]$$
$$\leq \frac{M_n^2\delta_n^2}{\delta^2}E_{P_X}\left[\sup_{\mathcal{N}_{osn}}\left\|\frac{d^2 m(X,\alpha)}{d\alpha^2}[u_n^*,u_n^*]\right\|_e^2\right] < \delta$$

eventually, which is satisfied given Assumption A.7(iii). The desired result follows. *Q.E.D.*

PROOF OF LEMMA A.5: For *Result (1)*, we first want to show that

$$\text{(C.21)}\quad \sup_{\mathcal{N}_{osn}}\left|\frac{1}{n}\sum_{i=1}^n\left\{\left\|\frac{d\widetilde{m}(X_i,\alpha)}{d\alpha}[u_n^*]\right\|^2_{\widehat{\Sigma}^{-1}} - \left\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|^2_{\Sigma^{-1}}\right\}\right|$$
$$\leq T_{n,\mathrm{I}} + T_{n,\mathrm{II}} + T_{n,\mathrm{III}} = o_{P_{Z^\infty}}(1),$$

where

$$T_{n,\mathrm{I}} = \sup_{\mathcal{N}_{osn}}\left|\frac{1}{n}\sum_{i=1}^n\left\{\left\|\frac{d\widetilde{m}(X_i,\alpha)}{d\alpha}[u_n^*]\right\|^2_{\widehat{\Sigma}^{-1}} - \left\|\frac{d\widetilde{m}(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|^2_{\widehat{\Sigma}^{-1}}\right\}\right|,$$
$$T_{n,\mathrm{II}} = \left|\frac{1}{n}\sum_{i=1}^n\left\{\left\|\frac{d\widetilde{m}(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|^2_{\widehat{\Sigma}^{-1}} - \left\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|^2_{\widehat{\Sigma}^{-1}}\right\}\right|,$$
$$T_{n,\mathrm{III}} = \left|\frac{1}{n}\sum_{i=1}^n\left\{\left\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|^2_{\widehat{\Sigma}^{-1}} - \left\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|^2_{\Sigma^{-1}}\right\}\right|.$$

Therefore, to prove equation (C.21), it suffices to show that

$$T_{n,j} = o_{P_{Z^\infty}}(1) \quad \text{for} \quad j \in \{\mathrm{I},\mathrm{II},\mathrm{III}\}.$$

Note that for $\|\cdot\|_{L^2(P_n)}$ with P_n being the empirical measure, $|\|a\|^2_{L^2(P_n)} - \|b\|^2_{L^2(P_n)}| \leq \|a-b\|^2_{L^2(P_n)} + 2|\langle b, a-b\rangle_{L^2(P_n)}|$. Now, let $a \equiv \frac{d\widetilde{m}(X_i,\alpha)}{d\alpha}[u_n^*]$ and $b \equiv \frac{d\widetilde{m}(X_i,\alpha_0)}{d\alpha}[u_n^*]$. In order to show $T_{n,\mathrm{I}} = o_{P_{Z^\infty}}(1)$, under Assumption 4.1(iii), it suffices to show

$$\sqrt{n^{-1}\sum_{i=1}^{n}\left\|\frac{d\widetilde{m}(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|_e^2} \times \sup_{\mathcal{N}_{osn}}\sqrt{n^{-1}\sum_{i=1}^{n}\left\|\frac{d\widetilde{m}(X_i,\alpha)}{d\alpha}[u_n^*]-\frac{d\widetilde{m}(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|_e^2}=o_{P_{Z^\infty}}(1).$$

By the property of LS projection, we have

$$n^{-1}\sum_{i=1}^{n}\left\|\frac{d\widetilde{m}(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|_e^2 \le n^{-1}\sum_{i=1}^{n}\left\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|_e^2 = O_{P_{Z^\infty}}(1)$$

due to i.i.d. data, Markov inequality, the definition of $E_{P_{Z^\infty}}[\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\|_{\Sigma^{-1}}^2]$, and Assumption 3.1(iv). Next, by the property of LS projection, we have

$$\begin{aligned}&\sup_{\mathcal{N}_{osn}} n^{-1}\sum_{i=1}^{n}\left\|\frac{d\widetilde{m}(X_i,\alpha)}{d\alpha}[u_n^*]-\frac{d\widetilde{m}(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|_e^2\\ &\quad\le \sup_{\mathcal{N}_{osn}} n^{-1}\sum_{i=1}^{n}\left\|\frac{dm(X_i,\alpha)}{d\alpha}[u_n^*]-\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|_e^2=o_{P_{Z^\infty}}(1)\end{aligned}$$

due to i.i.d. data, Markov inequality, and Assumption A.7(ii). Thus we established $T_{n,\mathrm{I}}=o_{P_{Z^\infty}}(1)$.

By similar algebra as before, in order to show $T_{n,\mathrm{II}}=o_{P_{Z^\infty}}(1)$, given Assumption 4.1(iii), it suffices to show

$$\sqrt{n^{-1}\sum_{i=1}^{n}\left\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|_e^2} \times \sqrt{n^{-1}\sum_{i=1}^{n}\left\|\frac{d\widetilde{m}(X_i,\alpha_0)}{d\alpha}[u_n^*]-\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|_e^2}=o_{P_{Z^\infty}}(1).$$

The term $n^{-1}\sum_{i=1}^{n}\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\|_e^2=O_{P_{Z^\infty}}(1)$ is due to i.i.d. data, Markov inequality, the definition of $E_{P_{Z^\infty}}[\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\|_{\Sigma^{-1}}^2]$, and Assumption 3.1(iv). The term $n^{-1}\sum_{i=1}^{n}\|\frac{d\widetilde{m}(X_i,\alpha_0)}{d\alpha}[u_n^*]-\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\|_e^2=o_{P_{Z^\infty}}(1)$ is due to i.i.d. data, Markov inequality, and Assumption A.6(i). Thus $T_{n,\mathrm{II}}=o_{P_{Z^\infty}}(1)$.

Finally, $T_{n,\mathrm{III}} = o_{P_{Z^\infty}}(1)$ follows from the fact that $n^{-1}\sum_{i=1}^{n}\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\|_e^2 = O_{P_{Z^\infty}}(1)$ and Assumption 4.1(iii). We thus established equation (C.21). Since

$$E_{P_{Z^\infty}}\left[n^{-1}\sum_{i=1}^{n}\left\|\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\right\|_{\Sigma^{-1}}^2\right] = E_{P_X}\left[g(X,u_n^*)\Sigma(X)g(X,u_n^*)'\right] \le C < \infty,$$

we obtain Result (1).

Result (2) immediately follows from equation (C.21) and Assumption B. *Q.E.D.*

APPENDIX D: SIEVE SCORE STATISTIC AND SCORE BOOTSTRAP

In the main text, we present the sieve Wald, SQLR statistics, and their bootstrap versions. Here we consider sieve score (or LM) statistic and its bootstrap version. Both the sieve score test and score bootstrap only require to compute the original-sample restricted PSMD estimator of α_0, and hence are computationally attractive.

Recall that $\widehat{\alpha}_n^R$ is the original-sample restricted PSMD estimator (4.10). Let $\widehat{v}_n^{*R}$ be computed in the same way as $\widehat{v}_n^*$ in Section 4.2, except that we use $\widehat{\alpha}_n^R$ instead of $\widehat{\alpha}_n$. And

$$\|\widehat{v}_n^{*R}\|_{n,\mathrm{sd}}^2 = n^{-1}\sum_{i=1}^{n}\left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n^R)}{d\alpha}[\widehat{v}_n^{*R}]\right)'\widehat{\Sigma}_i^{-1}\rho(Z_i,\widehat{\alpha}_n^R)\rho(Z_i,\widehat{\alpha}_n^R)'\widehat{\Sigma}_i^{-1} \times \left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n^R)}{d\alpha}[\widehat{v}_n^{*R}]\right).$$

Denote

$$\widehat{S}_n \equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n^R)}{d\alpha}[\widehat{v}_n^{*R}/\|\widehat{v}_n^{*R}\|_{n,\mathrm{sd}}]\right)'\widehat{\Sigma}_i^{-1}\widehat{m}(X_i,\widehat{\alpha}_n^R),$$

$$\widehat{S}_{1,n} \equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n^R)}{d\alpha}[\widehat{v}_n^{*R}/\|\widehat{v}_n^{*R}\|_{n,\mathrm{sd}}]\right)'\widehat{\Sigma}_i^{-1}\rho(Z_i,\widehat{\alpha}_n^R),$$

and

$$\widehat{S}_n^B \equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n^R)}{d\alpha}[\widehat{v}_n^{*R}/\|\widehat{v}_n^{*R}\|_{n,\mathrm{sd}}]\right)' \times \widehat{\Sigma}_i^{-1}\{\widehat{m}^B(X_i,\widehat{\alpha}_n^R) - \widehat{m}(X_i,\widehat{\alpha}_n^R)\},$$

$$\widehat{S}_{1,n}^B \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{d\widehat{m}(X_i, \widehat{\alpha}_n^R)}{d\alpha} [\widehat{v}_n^{*R} / \|\widehat{v}_n^{*R}\|_{n,\mathrm{sd}}] \right)' \times \widehat{\Sigma}_i^{-1} \{ (\omega_{i,n} - 1) \rho(Z_i, \widehat{\alpha}_n^R) \}.$$

Then

$$\begin{aligned} &\mathrm{Var}(\widehat{S}_{1,n}^B | Z^n) \\ &= \left(\sigma_\omega^2 \sum_{i=1}^n \left(\frac{d\widehat{m}(X_i, \widehat{\alpha}_n^R)}{d\alpha} [\widehat{v}_n^{*R}] \right)' \widehat{\Sigma}_i^{-1} \rho(Z_i, \widehat{\alpha}_n^R) \rho(Z_i, \widehat{\alpha}_n^R)' \widehat{\Sigma}_i^{-1} \right. \\ &\quad \left. \times \left(\frac{d\widehat{m}(X_i, \widehat{\alpha}_n^R)}{d\alpha} [\widehat{v}_n^{*R}] \right) \right) \Big/ (n \|\widehat{v}_n^{*R}\|_{n,\mathrm{sd}}^2) \\ &= \sigma_\omega^2, \end{aligned}$$

which coincides with that of $\widehat{S}_{1,n}$ (once adjusted by σ_ω^2).

Following the results in Section 4.2, one can compute $\widehat{v}_n^{*R}$ in closed form, $\widehat{v}_n^{*R} = \overline{\psi}^{k(n)}(\cdot)' \widetilde{D}_n^- \widetilde{F}_n$, where

$$\widetilde{F}_n = \frac{d\phi(\widehat{\alpha}_n^R)}{d\alpha} [\overline{\psi}^{k(n)}(\cdot)],$$

$$\widetilde{D}_n = n^{-1} \sum_{i=1}^n \left(\frac{d\widehat{m}(X_i, \widehat{\alpha}_n^R)}{d\alpha} [\overline{\psi}^{k(n)}(\cdot)'] \right)' \widehat{\Sigma}_i^{-1} \left(\frac{d\widehat{m}(X_i, \widehat{\alpha}_n^R)}{d\alpha} [\overline{\psi}^{k(n)}(\cdot)'] \right).$$

And $\|\widehat{v}_n^{*R}\|_{n,\mathrm{sd}}^2 = \widetilde{F}_n' \widetilde{D}_n^- \widetilde{\mho}_n \widetilde{D}_n^- \widetilde{F}_n$ with

$$\widetilde{\mho}_n = \frac{1}{n} \sum_{i=1}^n \left(\frac{d\widehat{m}(X_i, \widehat{\alpha}_n^R)}{d\alpha} [\overline{\psi}^{k(n)}(\cdot)'] \right)' \widehat{\Sigma}_i^{-1} \rho(Z_i, \widehat{\alpha}_n^R) \rho(Z_i, \widehat{\alpha}_n^R)' \widehat{\Sigma}_i^{-1} \times \left(\frac{d\widehat{m}(X_i, \widehat{\alpha}_n^R)}{d\alpha} [\overline{\psi}^{k(n)}(\cdot)'] \right).$$

Therefore, the bootstrap sieve score statistic $\widehat{S}_{1,n}^B$ can be expressed as

$$\begin{aligned} \widehat{S}_{1,n}^B &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{d\widehat{m}(X_i, \widehat{\alpha}_n^R)}{d\alpha} [\widehat{v}_n^{*R} / \|\widehat{v}_n^{*R}\|_{n,\mathrm{sd}}] \right)' \widehat{\Sigma}_i^{-1} (\omega_{i,n} - 1) \rho(Z_i, \widehat{\alpha}_n^R) \\ &= (\widetilde{F}_n' \widetilde{D}_n^- \widetilde{\mho}_n \widetilde{D}_n^- \widetilde{F}_n)^{-1/2} \widetilde{F}_n' \widetilde{D}_n^- \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{d\widehat{m}(X_i, \widehat{\alpha}_n^R)}{d\alpha} [\overline{\psi}^{k(n)}(\cdot)'] \right)' \\ &\quad \times \widehat{\Sigma}_i^{-1} (\omega_{i,n} - 1) \rho(Z_i, \widehat{\alpha}_n^R). \end{aligned}$$

For the case of i.i.d. weights, this expression is similar to that proposed in Kline and Santos (2012) for parametric models, which suggests the potential higher order refinements of the bootstrap sieve score test $(\widehat{S}^B_{1,n})^2$. We leave it to future research for bootstrap refinement.

In the rest of this section, to simplify presentation, we assume that $\widehat{m}(x,\alpha)$ is a series LS estimator (2.5) of $m(x,\alpha)$. Then we have

$$\begin{aligned}
&\widehat{m}^B\bigl(x,\widehat{\alpha}^R_n\bigr)-\widehat{m}\bigl(x,\widehat{\alpha}^R_n\bigr)\\
&\quad=\left(\sum_{j=1}^n(\omega_{j,n}-1)\rho\bigl(Z_j,\widehat{\alpha}^R_n\bigr)p^{J_n}(X_j)'\right)\bigl(P'P\bigr)^{-}p^{J_n}(x).
\end{aligned}$$

When $\widehat{\Sigma}=I$, then we have

$$\begin{aligned}
\widehat{S}_n&=\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(\frac{d\widehat{m}\bigl(X_i,\widehat{\alpha}^R_n\bigr)}{d\alpha}\bigl[\widehat{v}^{*R}_n/\bigl\|\widehat{v}^{*R}_n\bigr\|_{n,\mathrm{sd}}\bigr]\right)'\rho\bigl(Z_i,\widehat{\alpha}^R_n\bigr)=\widehat{S}_{1,n},\\
\widehat{S}^B_n&=\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(\frac{d\widehat{m}\bigl(X_i,\widehat{\alpha}^R_n\bigr)}{d\alpha}\bigl[\widehat{v}^{*R}_n/\bigl\|\widehat{v}^{*R}_n\bigr\|_{n,\mathrm{sd}}\bigr]\right)'(\omega_{i,n}-1)\rho\bigl(Z_i,\widehat{\alpha}^R_n\bigr)\\
&=\widehat{S}^B_{1,n}.
\end{aligned}$$

Let $\{\epsilon_n\}_{n=1}^\infty$ and $\{\zeta_n\}_{n=1}^\infty$ be real-valued positive sequences such that $\epsilon_n=o(1)$ and $\zeta_n=o(1)$.

ASSUMPTION D.1: (i) $\max\{\epsilon_n,n^{-1/4}\}M_n\delta_n=o(n^{-1/2})$

$$\begin{aligned}
&\sup_{\mathcal{N}_{osn}}\sup_{u\in\overline{\mathbf{V}}_n:\|u\|=1}n^{-1}\sum_{i=1}^n\left\|\frac{d\widehat{m}(X_i,\alpha)}{d\alpha}[u]-\frac{dm(X_i,\alpha)}{d\alpha}[u]\right\|_e^2\\
&\quad=O_{P_{Z^\infty}}\bigl(\max\bigl\{n^{-1/2},\epsilon_n^2\bigr\}\bigr);
\end{aligned}$$

(ii) *there is a continuous mapping* $\Upsilon:\mathbb{R}_+\to\mathbb{R}_+$ *such that* $\max\{\Upsilon(\zeta_n),n^{-1/4}\}M_n\delta_n=o(n^{-1/2})$ *and*

$$\begin{aligned}
&\sup_{\mathcal{N}_{osn}}\sup_{\overline{\mathbf{V}}_n:\|u^*_n-u\|\le\zeta_n}n^{-1}\sum_{i=1}^n\left\|\frac{dm(X_i,\alpha)}{d\alpha}\bigl[u^*_n\bigr]-\frac{dm(X_i,\alpha)}{d\alpha}[u]\right\|_e^2\\
&\quad=O_{P_{Z^\infty}}\bigl(\max\bigl\{n^{-1/2},\bigl(\Upsilon(\zeta_n)\bigr)^2\bigr\}\bigr);
\end{aligned}$$

(iii) $\|\widehat{u}^{*R}_n-u^*_n\|=O_{P_{Z^\infty}}(\zeta_n)$ *where* $\widehat{u}^{*R}_n\equiv\widehat{v}^{*R}_n/\|\widehat{v}^{*R}_n\|_{\mathrm{sd}}$.

Assumption D.1(i) can be obtained by similar conditions to those imposed in Ai and Chen (2003). Assumption D.1(ii) can be established by controlling the entropy, as in VdV-W, Chapter 2.11 and $E[\|\frac{dm(X,\alpha)}{d\alpha}[u_n^*]-\frac{dm(X,\alpha)}{d\alpha}[u]\|_e^2]=o(1)$ for all $\|u_n^*-u\|<\zeta_n$; this result is akin to that in Lemma 1 of Chen, Linton, and van Keilegom (2003). However, Assumption D.1(ii) can also be obtained by weaker conditions, yielding a $(Y(\zeta_n))^2$ that is slower than $O(n^{-1/2})$ provided that $Y(\zeta_n)M_n\delta_n=o(n^{-1/2})$. In the proof, we show that $\|\widehat{u}_n^{*R}-u_n^*\|=o_{P_{Z^\infty}}(1)$; faster rates of convergence will relax the conditions needed to show part (ii).

THEOREM D.1: *Let $\widehat{\alpha}_n^R$ be the restricted PSMD estimator* (4.10), *and conditions for Lemma* 3.2 *and Proposition* B.1 *hold*. *Let Assumptions* 3.5, A.4–A.7, 3.6(ii), 4.1, B.1, *and* D.1 *hold and that $n\delta_n^2(M_n\delta_{s,n})^{2\kappa}C_n=o(1)$. Then*, *under the null hypothesis of $\phi(\alpha_0)=\phi_0$*,

(1) $\widehat{S}_n=\sqrt{n}\mathbb{Z}_n+o_{P_{Z^\infty}}(1)\Rightarrow N(0,1)$.

(2) *Further*, *if conditions for Lemma* A.1 *and Assumptions* Boot.3(ii), Boot.1, *or* Boot.2 *hold*, *then*

$$\left|\mathcal{L}_{V^\infty|Z^\infty}\left(\sigma_\omega^{-1}\widehat{S}_n^B|Z^n\right)-\mathcal{L}(\widehat{S}_n)\right|=o_{P_{Z^\infty}}(1),\quad \textit{and}$$

$$\sup_{t\in\mathbb{R}}\left|P_{V^\infty|Z^\infty}\left(\sigma_\omega^{-1}\widehat{S}_n^B\le t|Z^n\right)-P_{Z^\infty}(\widehat{S}_n\le t)\right|$$
$$=o_{P_{V^\infty|Z^\infty}}(1)\quad wpa1\ (P_{Z^\infty}).$$

PROOF: We first note that by Lemma 5.1, Assumptions 3.6(i) and Boot.3(i) hold. Also, by Proposition B.1 we have $\widehat{\alpha}_n^R\in\mathcal{N}_{osn}$ wpa1 under the null hypothesis of $\phi(\alpha_0)=\phi_0$. Under the null hypothesis, and Assumption 3.5, we also have (see Step 1 in the proof of Theorem 4.3):

$$\sqrt{n}\langle u_n^*,\widehat{\alpha}_n^R-\alpha_0\rangle=o_{P_{Z^\infty}}(1).$$

For *Result (1)*, we show that $\widehat{S}_n$ is asymptotically standard normal under the null hypothesis in two steps.

STEP 1: We first show that $|\frac{\|\widehat{v}_n^{*R}\|_{\mathrm{sd}}}{\|\widehat{v}_n^{*R}\|_{n,\mathrm{sd}}}-1|=o_{P_{Z^\infty}}(1)$ and $\|\widehat{u}_n^{*R}-u_n^*\|=o_{P_{Z^\infty}}(1)$, where $\widehat{u}_n^{*R}\equiv\widehat{v}_n^{*R}/\|\widehat{v}_n^{*R}\|_{\mathrm{sd}}$ and $\widehat{v}_n^{*R}$ is computed in the same way as that in Section 4.2, except that we use $\widehat{\alpha}_n^R$ instead of $\widehat{\alpha}_n$.

$|\frac{\|\widehat{v}_n^{*R}\|_{\mathrm{sd}}}{\|\widehat{v}_n^{*R}\|_{n,\mathrm{sd}}}-1|=o_{P_{Z^\infty}}(1)$ can be established in the same way as that of Theorem 4.2(1). Also, following the proof of Theorem 4.2(1), we obtain

$$\left\|\frac{\widehat{v}_n^{*R}-v_n^*}{\|v_n^*\|}\right\|=o_{P_{Z^\infty}}(1),\qquad \frac{\|\widehat{v}_n^{*R}\|}{\|v_n^*\|_{\mathrm{sd}}}=O_{P_{Z^\infty}}(1),$$

$$\sup_{v\in\overline{\mathbf{V}}_n}\left|\frac{\langle v_n^*-\widehat{v}_n^{*R},v\rangle}{\|v\|\times\|\widehat{v}_n^{*R}\|}\right|=o_{P_{Z^\infty}}(1).$$

This and Assumption 3.1(iv) imply that $|\frac{\langle \widehat{v}_n^{*R}, \widehat{v}_n^{*R}-v_n^*\rangle}{\|\widehat{v}_n^{*R}\|_{\rm sd}^2}| = o_{P_{Z^\infty}}(1)$ and $|\frac{\langle v_n^*, \widehat{v}_n^{*R}-v_n^*\rangle}{\|\widehat{v}_n^{*R}\|_{\rm sd}^2}| = \frac{\|v_n^*\|_{\rm sd}}{\|\widehat{v}_n^{*R}\|_{\rm sd}} \times o_{P_{Z^\infty}}(1)$. Therefore,

$$\left|\frac{\|v_n^*\|_{\rm sd}^2}{\|\widehat{v}_n^{*R}\|_{\rm sd}^2} - 1\right| \le \left|\frac{\langle \widehat{v}_n^{*R}, \widehat{v}_n^{*R}-v_n^*\rangle}{\|\widehat{v}_n^{*R}\|_{\rm sd}^2}\right| + \left|\frac{\langle v_n^*, \widehat{v}_n^{*R}-v_n^*\rangle}{\|\widehat{v}_n^{*R}\|_{\rm sd}^2}\right| = o_{P_{Z^\infty}}(1),$$

and

$$\left|\frac{\|v_n^*\|_{\rm sd}}{\|\widehat{v}_n^{*R}\|_{\rm sd}} - 1\right| = o_{P_{Z^\infty}}(1).$$

Thus

$$\begin{aligned}\|\widehat{u}_n^{*R} - u_n^*\| &= \left\|\frac{\widehat{v}_n^{*R}}{\|\widehat{v}_n^{*R}\|_{\rm sd}} - \frac{v_n^*}{\|v_n^*\|_{\rm sd}}\right\| = \left\|\frac{\widehat{v}_n^{*R}}{\|v_n^*\|_{\rm sd}}\big(1+o_{P_{Z^\infty}}(1)\big) - \frac{v_n^*}{\|v_n^*\|_{\rm sd}}\right\| \\ &= \left\|\frac{\widehat{v}_n^{*R}-v_n^*}{\|v_n^*\|_{\rm sd}}\right\| + o_{P_{Z^\infty}}\left(\frac{\|\widehat{v}_n^{*R}\|}{\|v_n^*\|_{\rm sd}}\right) = o_{P_{Z^\infty}}(1).\end{aligned}$$

STEP 2: We show that under the null hypothesis,

$$\begin{aligned}\widehat{S}_n &= \sqrt{n}\mathbb{Z}_n + o_{P_{Z^\infty}}(1) \\ &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^n \left(\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\right)' \Sigma(X_i)^{-1}\rho(Z_i,\alpha_0) + o_{P_{Z^\infty}}(1).\end{aligned} \tag{D.1}$$

By Step 1, it suffices to show that under the null hypothesis,

$$\begin{aligned}\overline{S}_n &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^n \left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n^R)}{d\alpha}[\widehat{u}_n^{*R}]\right)' \widehat{\Sigma}^{-1}(X_i)\widehat{m}(X_i,\widehat{\alpha}_n^R) \\ &= \sqrt{n}\mathbb{Z}_n + o_{P_{Z^\infty}}(1).\end{aligned}$$

Recall that $\ell_n(x,\alpha) \equiv \widehat{m}(x,\alpha_0) + \widetilde{m}(x,\alpha)$. We have

$$\begin{aligned}&\left|\overline{S}_n - \frac{1}{\sqrt{n}}\sum_{i=1}^n \left(\frac{d\widehat{m}(X_i,\widehat{\alpha}_n^R)}{d\alpha}[\widehat{u}_n^{*R}]\right)' \widehat{\Sigma}(X_i)^{-1}\ell_n(X_i,\widehat{\alpha}_n^R)\right| \\ &\quad \le \sqrt{n}\sqrt{n^{-1}\sum_{i=1}^n \left\|\widehat{\Sigma}^{-1/2}(X_i)\frac{d\widehat{m}(X_i,\widehat{\alpha}_n^R)}{d\alpha}[\widehat{u}_n^{*R}]\right\|_e^2} \\ &\qquad \times \sqrt{n^{-1}\sum_{i=1}^n \|\widehat{m}(X_i,\widehat{\alpha}_n^R) - \ell_n(X_i,\widehat{\alpha}_n^R)\|_e^2}.\end{aligned}$$

By Lemma A.2(1) and the assumption that $n\delta_n^2(M_n\delta_{s,n})^{2\kappa}C_n = o(1)$, we have

$$\sqrt{n^{-1}\sum_{i=1}^{n}\left\|\widehat{m}\left(X_i,\widehat{\alpha}_n^R\right)-\ell_n\left(X_i,\widehat{\alpha}_n^R\right)\right\|_e^2} = o_{P_{Z^\infty}}\left(n^{-1/2}\right).$$

Also $n^{-1}\sum_{i=1}^{n}\|\widehat{\Sigma}^{-1/2}(X_i)\frac{d\widehat{m}(X_i,\widehat{\alpha}_n^R)}{d\alpha}[\widehat{u}_n^{*R}]\|_e^2 \asymp 1$ by Step 1 and Assumptions A.7 and D.1. Therefore

$$\overline{S}_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{d\widehat{m}\left(X_i,\widehat{\alpha}_n^R\right)}{d\alpha}\left[\widehat{u}_n^{*R}\right]\right)'\widehat{\Sigma}(X_i)^{-1}\ell_n\left(X_i,\widehat{\alpha}_n^R\right) + o_{P_{Z^\infty}}(1).$$

Assumption D.1(i) implies that

$$n^{-1}\sum_{i=1}^{n}\left\|\frac{d\widehat{m}\left(X_i,\widehat{\alpha}_n^R\right)}{d\alpha}\left[\widehat{u}_n^{*R}\right]-\frac{dm\left(X_i,\widehat{\alpha}_n^R\right)}{d\alpha}\left[\widehat{u}_n^{*R}\right]\right\|_e^2$$
$$= O_{P_{Z^\infty}}\left(\max\left\{n^{-1/2},\epsilon_n^2\right\}\right).$$

And $n^{-1}\sum_{i=1}^{n}\|\ell_n(X_i,\widehat{\alpha}_n^R)\|_e^2 = O_{P_{Z^\infty}}((M_n\delta_n)^2)$ by Lemma A.2(2). These results, Assumption D.1(i), and Assumption 4.1(iii) together lead to

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{d\widehat{m}\left(X_i,\widehat{\alpha}_n^R\right)}{d\alpha}\left[\widehat{u}_n^{*R}\right]\right)'\widehat{\Sigma}(X_i)^{-1}\ell_n\left(X_i,\widehat{\alpha}_n^R\right)$$
$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{dm\left(X_i,\widehat{\alpha}_n^R\right)}{d\alpha}\left[\widehat{u}_n^{*R}\right]\right)'\Sigma(X_i)^{-1}\ell_n\left(X_i,\widehat{\alpha}_n^R\right) + o_{P_{Z^\infty}}(1)$$
$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{dm\left(X_i,\widehat{\alpha}_n^R\right)}{d\alpha}\left[u_n^*\right]\right)'\Sigma(X_i)^{-1}\ell_n\left(X_i,\widehat{\alpha}_n^R\right) + o_{P_{Z^\infty}}(1),$$

where the second equality is due to $\|\widehat{u}_n^{*R}-u_n^*\| = O_{P_{Z^\infty}}(\zeta_n)$ (Assumption D.1(iii)) and Assumption D.1(ii).

Since $\widehat{\alpha}_n^R \in \mathcal{N}_{osn}$ wpa1 under the null hypothesis, $\sqrt{n}\langle u_n^*, \widehat{\alpha}_n^R - \alpha_0\rangle = o_{P_{Z^\infty}}(1)$, and by analogous calculations to those in the proof of Lemma A.3, we obtain

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{dm\left(X_i,\widehat{\alpha}_n^R\right)}{d\alpha}\left[u_n^*\right]\right)'\Sigma(X_i)^{-1}\ell_n\left(X_i,\widehat{\alpha}_n^R\right) = \sqrt{n}\mathbb{Z}_n + o_{P_{Z^\infty}}(1),$$

and hence equation (D.1) holds. By Assumption 3.6(ii), we have: $\widehat{S}_n \Rightarrow N(0,1)$ under the null hypothesis.

For *Result (2)*, we now show that $\widehat{S}_n^B$ also converges weakly (in the sense of Bootstrap Section 5) to a standard normal under the null hypothesis. It suffices to show that

$$\widehat{S}_n^B = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\omega_i - 1)\left(\frac{dm(X_i,\alpha_0)}{d\alpha}[u_n^*]\right)' \Sigma(X_i)^{-1}\rho(Z_i,\alpha_0) + o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{Z^\infty}). \tag{D.2}$$

Note that $\ell_n^B(X_i,\widehat{\alpha}_n^R) - \ell_n(X_i,\widehat{\alpha}_n^R) = \widehat{m}^B(X_i,\alpha_0) - \widehat{m}(X_i,\alpha_0)$, and that $n^{-1}\sum_{i=1}^n \|\widehat{m}^B(X_i,\alpha_0) - \widehat{m}(X_i,\alpha_0)\|_e^2 = O_{P_{V^\infty|Z^\infty}}(J_n/n)$ wpa1 (P_{Z^∞}) (see the proof of Lemma A.2). We have, by calculations similar to Step 2,

$$\left|\widehat{S}_n^B - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{dm(X_i,\widehat{\alpha}_n^R)}{d\alpha}[u_n^*]\right)' \Sigma(X_i)^{-1}\{\ell_n^B(X_i,\widehat{\alpha}_n^R) - \ell_n(X_i,\widehat{\alpha}_n^R)\}\right| = o_{P_{V^\infty|Z^\infty}}(1) \quad \text{wpa1 } (P_{Z^\infty}).$$

By analogous calculations to those in the proof of Lemma A.3, we obtain equation (D.2). This and Result (1) and Assumption Boot.3(ii) now imply that under the null and conditional on the data, $\sigma_\omega^{-1}\widehat{S}_n^B$ is also asymptotically standard normally distributed. The last part of Result (2) can be established in the same way as that of Theorem 5.2(1), and is omitted. *Q.E.D.*

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