SUPPLEMENT TO "DYNAMIC NOISY RATIONAL EXPECTATIONS EQUILIBRIUM WITH INSIDER INFORMATION"

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Abstract. This supplementary document contains the proofs of many results in "Dynamic Noisy Rational Expectations Equilibrium with Insider Information", as well as both statements and proofs of a number of technical Lemmas.

1. Lemmas Regarding Θ

We first prove three lemmas regarding Θ from $\{(4.1)\}$[1], valid under Assumption $\{2.1\}$. For $M \in \mathbb{S}^d_{++}$ write $0 < \underline{M} < \overline{M}$ as the smallest and largest eigenvalues. Additionally, the bounding constant $\check{K} > 0$ below may change from line to line.

Lemma 1.1. *There exists $\hat{K} > 0$ such that for $x \in E, y \in \mathbb{R}^d$, (1) $\Theta(x,y) \geq -\hat{K}\left(1 + y'y\right)$ and (2) $|\Theta(x,y)| \leq \hat{K}\left(1 + x'x + y'y\right)$.*

Proof of Lemma 1.1. Let $0 < \delta < \underline{M}$. By part (e) of Assumption $\{2.1\}$:

$$\Theta(x,y) \geq -K_2\left(1 + |x|^{2-\varepsilon_1}\right) + \frac{1}{2}(\underline{M} - \delta)|x|^2 + \frac{1}{2}\delta x'x + x'\zeta - x'\tilde{M}y.$$

Clearly, $-K_2(1 + |x|^{2-\varepsilon_1}) + (1/2)(\underline{M} - \delta)|x|^2 \geq -\check{K}$. As $\delta x'x - 2x'(\tilde{M}y - \zeta) \geq -(1/\delta)(\tilde{M}y-\zeta)'(\tilde{M}-\zeta)$, the Cauchy-Schwartz inequality implies this may further be bounded from below by $-(2\overline{\tilde{M}'\tilde{M}}/\delta)\times y'y - (2/\delta)\zeta'\zeta$. (1) now readily follows. Part (2) is obvious from Assumption $\{2.1\}$ part (d), and the Cauchy-Schwartz inequality. $\square$

Lemma 1.2. *The maps $y \to e^{-\gamma\Theta(X_T,y)}$ and $y \to \Psi(X_T)e^{-\gamma\Theta(X_T,y)}$ are analytic from $\mathbb{R}^d$ to $L^1(\mathbb{R})$ and $L^1(\mathbb{R}^d)$ respectively.*

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[1]References to the main are surrounded by curly brackets.

Proof of Lemma 1.2. Part (1) of Lemma 1.1 implies $e^{-\gamma\Theta(X_T,y)} \in L^1(\mathbb{R})$ for all $y \in \mathbb{R}^d$. Next, using the multi-dimensional Taylor theorem, for any fixed $g \in \mathbb{R}^d$ we may write

$$e^{-\gamma\theta(x,y)} = e^{-\gamma\left(\Pi'\Psi(x)+\frac{1}{2}x'Mx+x'\zeta\right)}e^{\gamma y'\tilde{M}x} = \sum_\alpha A_\alpha(x,g)(y-g)^\alpha,$$

where $\alpha = (\alpha_1, ..., \alpha_d)$ is a multi-index of non-negative integers, $|\alpha| = \alpha_1 + ... + \alpha_d$, $(y-g)^\alpha = \prod_{i=1}^d (y_i - g_i)^{\alpha_i}$, and

$$A_\alpha(x,g) = e^{-\gamma\left(\Pi'\Psi(x)+\frac{1}{2}x'Mx+x'\zeta\right)}\frac{1}{|\alpha|!}D_\alpha\left(e^{\gamma y'\tilde{M}x}\right)\Big|_{y=g} = e^{-\gamma\Theta(x,g)}\frac{1}{|\alpha|!}\gamma^{|\alpha|}(\tilde{M}x)^\alpha.$$

Above, $D_\alpha f$ is the $|\alpha|^{th}$ partial derivative of f, α_i times with respect to y_i for $i = 1, ..., d$. Since $|x^\alpha| \le |x|^{|\alpha|}$ and $|\tilde{M}x| \le \overline{\tilde{M}}|x|$ it follows that

$$|(\tilde{M}x)^\alpha| \le |\tilde{M}x|^{|\alpha|} \le (\overline{\tilde{M}})^{|\alpha|}|x|^{|\alpha|}.$$

Let ε_0 be from Assumption {2.1} and write $M_* = \gamma\overline{\tilde{M}}/(\varepsilon_0)$. If $|\alpha| = n$ then by part (1) of Lemma 1.1 and the bound $|x|^n \le e^{\varepsilon_0|x|}(n/\varepsilon_0)^n e^{-n}$

$$\mathbb{E}\left[|A_\alpha(X_T,g)|\right] \le e^{\gamma\hat{K}(1+g'g)}\frac{(nM_*)^n}{n!e^n}\mathbb{E}\left[e^{\varepsilon_0|X_T|}\right] \le \check{K}e^{\gamma\hat{K}(1+g'g)}\frac{(M_*)^n}{\sqrt{n}}\mathbb{E}\left[e^{\varepsilon_0|X_T|}\right], \tag{1.1}$$

where the last inequality follows from Stirling's formula. Thus, if there is $L > 1$ such that $\max_{i=1,\ldots,d}|y_i - g_i| \le 1/(LM_*)$, then for $N = 1, 2, \ldots$:

$$\begin{aligned}
\mathbb{E}\left[\left|\sum_{n\ge N}\sum_{|\alpha|=n}A_\alpha(X_T,g)(y-g)^\alpha\right|\right] &\le \sum_{n\ge N}\sum_{|\alpha|=n}\mathbb{E}\left[|A_\alpha(X_T,g)|\right]\prod_{i=1}^d|y_i-g_i|^{\alpha_i} \\
&\le \check{K}e^{\gamma\hat{K}(1+g'g)}\mathbb{E}\left[e^{\varepsilon_0|X_T|}\right]\sum_{n\ge N}\sum_{|\alpha|=n}\frac{1}{\sqrt{n}}L^{-n}; \\
&= \check{K}e^{\gamma\hat{K}(1+g'g)}\mathbb{E}\left[e^{\varepsilon_0|X_T|}\right]\sum_{n\ge N}\frac{1}{\sqrt{n}}L^{-n}\begin{pmatrix} n+d-1 \\ n\end{pmatrix}; \\
&\le \check{K}e^{\gamma\hat{K}(1+g'g)}\mathbb{E}\left[e^{\varepsilon_0|X_T|}\right]\sum_{n\ge N}L^{-n}n^{d-\frac{1}{2}}.
\end{aligned}$$

The right hand side of the last inequality goes to 0 as $N \uparrow \infty$ which proves $y \to e^{-\gamma\Theta(X_T,y)}$ is an analytic map from $\mathbb{R}^d$ to $L^1(\mathbb{R})$. We next consider $y \to \Psi(X_T)e^{-\gamma\Theta(X_T,y)}$. As $\Psi(X_T)$ does not depend upon y, the proof is very similar and we only show the differences. First, that $\Psi(X_T)e^{-\gamma\Theta(X_T,y)} \in L^1(\mathbb{R}^d)$ follows from Assumption {2.1} and part (1) of Lemma 1.1, since

$$|\Psi(X_T)|e^{-\gamma\Theta(X_T,y)} \le K_1(1 + X_T'X_T)e^{\gamma\hat{K}(1+y'y)} \le \check{K}e^{\varepsilon_0|X_T|}e^{\hat{K}(1+y'y)}.$$

where the last inequality uses the estimate

$$x^2 \leq \frac{2}{k^2}\left(e^{kx}-1\right) \leq \frac{2}{k^2}e^{kx}; \qquad x,k>0. \tag{1.2}$$

Next, the analytic convergence proof is the same except in (1.1) the first line (right hand side) should have $|\Psi(X_T)|$ within the expected value. Then, going from the first to the second line we use, for $\varepsilon_2, \delta > 0$ such that $\delta + \varepsilon_2 < \varepsilon_0$:

$$|\Psi(X_T)||X_T|^n \leq K_1(1+X_T'X_T)e^{\delta|X_T|}\left(\frac{n}{\delta}\right)^n e^{-n} \leq \check{K}e^{\varepsilon_0|X_T|}\left(\frac{n}{\delta}\right)^n e^{-n}.$$

From here, the rest of the proof is the same. □

Lemma 1.3. *There exists a constant $\hat{C}$ so that*

$$\frac{\mathbb{E}\left[X_T'X_Te^{-\gamma\Theta(X_T,y)}\right]}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,y)}\right]} \leq \hat{C}\left(1+\mathbb{E}\left[e^{\varepsilon_0|X_T|}\right]+y'y\right).$$

Proof of Lemma 1.3. For $\tau \geq 0$, define $f(\tau) := -(1/\gamma)\log\left(\mathbb{E}\left[e^{-\gamma\tau\Theta(X_T,y)}\right]\right)$. By part (1) of Lemma 1.1, $f(\tau) \geq \hat{K}(1+y'y)\tau$. By Jensen's inequality, part (2) of Lemma 1.1, and (1.2) we deduce $f(\tau) \leq \check{K}(1+y'y+\mathbb{E}\left[e^{\varepsilon_0|X_T|}\right])\tau$, and hence f is linearly bounded. The dominated convergence theorem shows f is smooth (c.f (Dembo and Zeitouni, 1998, Lemma 2.5)), and Hölder's inequality shows f is concave. Therefore

$$\dot{f}(1) \leq \lim_{\varepsilon\downarrow 0}\dot{f}(\varepsilon) = \lim_{\varepsilon\downarrow 0}\frac{\mathbb{E}\left[\Theta(X_T,y)e^{-\gamma\varepsilon\Theta(X_T,y)}\right]}{\mathbb{E}\left[e^{-\gamma\varepsilon\Theta(X_T,y)}\right]}.$$

Part (1) of Lemma 1.1 readily implies the uniform integrability of $\left\{e^{-\gamma\varepsilon\Theta(X_T,y)}\right\}_{\varepsilon>0}$. Parts (1),(2) of Lemma 1.1, along with (1.2) also imply, for $\varepsilon < 1$

$$\begin{aligned}|\Theta(x,y)|e^{-\gamma\varepsilon\Theta(x,y)} &= \Theta(x,y)^+e^{-\gamma\varepsilon\Theta(x,y)}1_{\Theta(x,y)\geq 0} + \Theta(x,y)^-e^{-\gamma\varepsilon\Theta(x,y)}1_{\Theta(x,y)<0};\\ &\leq |\Theta(x,y)| + \hat{K}(1+y'y)e^{\gamma\hat{K}(1+y'y)};\\ &\leq \check{K}(y'y+e^{\varepsilon_0|x|}) + \hat{K}(1+y'y)e^{\gamma\hat{K}(1+y'y)}.\end{aligned}$$

Thus, by Assumption {2.1}, part (c) and dominated convergence we conclude

$$\dot{f}(1) = \frac{\mathbb{E}\left[\Theta(X_T,y)e^{-\gamma\Theta(X_T,y)}\right]}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,y)}\right]} \leq \mathbb{E}\left[\Theta(X_T,y)\right].$$

Plugging in for Θ gives

$$\begin{aligned}&\frac{\mathbb{E}\left[X_T'MX_Te^{-\gamma\Theta(X_T,y)}\right]}{2\mathbb{E}\left[e^{-\gamma\Theta(X_T,y)}\right]}\\ &\qquad\leq \mathbb{E}\left[|\Theta(X_T,y)|\right] - \frac{\mathbb{E}\left[\left(\Pi'\Psi(X_T)+X_T'\zeta-(X_T)'\tilde{M}y\right)e^{-\gamma\Theta(X_T,y)}\right]}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,y)}\right]}.\end{aligned} \tag{1.3}$$

We claim Assumption {2.1}(e) implies for $\delta > 0$ there exists a constant $\check{K}$ so that

$$\Pi'\Psi(x) + x'\zeta - x'\tilde{M}y \geq -\check{K}(1 + y'y) - \frac{\delta}{2}x'Mx. \tag{1.4}$$

Admitting this, taking $\delta = 1/2$ and using (1.3) we deduce

$$\frac{1}{4}\frac{\mathbb{E}\left[X_T'MX_Te^{-\gamma\Theta(X_T,y)}\right]}{\mathbb{E}\left[e^{-\gamma\Theta(X_T,y)}\right]} \leq \check{K}(1 + y'y) + \mathbb{E}\left[|\Theta(X_T,y)|\right] \leq \check{K}\left(1 + y'y + \mathbb{E}\left[e^{\varepsilon_0|X_T|}\right]\right),$$

where the last inequality follows from Lemma 1.1 and (1.2). The result holds since $x'x \leq (1/\underline{\mathrm{M}})x'Mx$. It thus remains to prove (1.4). First, for $\delta > 0$

$$\begin{aligned}\Pi'\Psi(x) &= \Pi'\Psi(x) \pm \frac{\delta}{2}x'Mx \geq -K_2(1 + |x|^{2-\varepsilon_1}) + \frac{\delta\underline{M}}{2T}x'x - \frac{\delta}{2}x'Mx;\\ &\geq -\check{K}(\delta) - \frac{\delta}{2}x'Mx.\end{aligned}$$

A similar calculation gives a commensurate lower bound for $x'\zeta$. (1.4) follows as

$$-x'\tilde{M}y \geq -x'\tilde{M}y + \frac{\delta\underline{M}}{2}x'x - \frac{\delta}{2}x'Mx \geq -\frac{\overline{\tilde{M}'\tilde{M}}}{2\underline{M}^2\delta^2}y'y - \frac{\delta}{2}x'Mx.$$

□

2. Lemmas Regarding the Full Communication Equilibrium

Throughout, we enforce Assumptions {2.1}, {A.1}. Recall u from {(4.5)} and {(B.2)}, and G_I from {(2.5)}. The following lemma shows u governs the conditional laws of G_I given $\mathbb{F}^B$, as well as the Brownian motion B^m under $\mathbb{F}^B \vee \sigma(G_I)$.

Lemma 2.1.

(1) For each $t \leq T$, the law of G_I given $\mathcal{F}_t^B$ has pdf $u(t, X_t, \cdot)$. Therefore, $\mathbb{P}\left[G_I \in \cdot \middle| \mathcal{F}_t^B\right] \sim \mathbb{P}\left[G_I \in \cdot\right]$ almost surely, with density

$$\tilde{p}_t^g := \frac{u(t, X_t, g)}{u(0, X_0, g)} = \mathcal{E}\left(\int_0^\cdot (\tilde{\mu}_u^g)'dB_u\right)_t; \quad \tilde{\mu}_t^g := a(X_t)'\nabla_x\left(\log\left(u(t, X_t, g)\right)\right). \tag{2.1}$$

(2) $\mathbb{F}^m = \mathbb{F}^B \vee \sigma(G_I)$ is right-continuous, $1/\tilde{p}^{G_I}$ is a $(\mathbb{P}, \mathbb{F}^m)$-martingale, and the martingale preserving measure takes the form

$$\frac{d\widetilde{\mathbb{P}}^{G_I}}{d\mathbb{P}} = \frac{1}{\tilde{p}_T^{G_I}}; \quad \tilde{p}_\cdot^{G_I} = \mathcal{E}\left(\int_0^\cdot (\tilde{\mu}_u^{G_I})'dB_u\right)_\cdot. \tag{2.2}$$

(3) B is a $(\widetilde{\mathbb{P}}^{G_I}, \mathbb{F}^m)$ Brownian motion with the predictable representation property (PRP), and $B_\cdot^m := B_\cdot - \int_0^\cdot \tilde{\mu}_u^{G_I}du$, is a $(\mathbb{P}, \mathbb{F}^m)$-Brownian motion on $[0, T]$ with the PRP.

Proof of Lemma 2.1. Let $\phi \in C_c^\infty(\mathbb{R}^d)$, $t \leq T$ and set $Y^I = 1/\sqrt{T}\sqrt{C_I}W^I$ for a d-dimensional Brownian motion independent of B, and note that Y^I is a Markov process with transition kernel

$$p_{C_I}(\tau, x, y) = \frac{1}{(2\pi)^{d/2}\sqrt{|C_I|}}\sqrt{\frac{T}{\tau}}e^{-\frac{T}{2\tau}(y-x)'C_I^{-1}(y-x)}; \quad \tau > 0, x, y \in \mathbb{R}^d.$$

Similarly to {(7.8)} set $\hat{p}_{C_I}(y) = p_{C_I}(T, 0, y)$. By the tower property

$$\mathbb{E}\left[\phi(G_N)\middle|\mathcal{F}_t^B\right] = \mathbb{E}\left[\mathbb{E}\left[\phi(X_T + Y_T^I)\middle|\mathcal{F}_t^{B,W^I}\right]\middle|\mathcal{F}_t^B\right].$$

Using the Markov property

$$\mathbb{E}\left[\phi(X_T + Y_T^I)\middle|\mathcal{F}_t^{B,W^I}\right] = \int \phi(x+y)p(T-t, X_t, x)p_{C_I}(T-t, Y_t^I, y)dxdy,$$

where the integration region is $E \times \mathbb{R}^d$. Therefore, by the independence of B and Y^I

$$\begin{aligned}
\mathbb{E}\left[\phi(G_N)\middle|\mathcal{F}_t^B\right] &= \int \phi(x+y)p(T-t, X_t, x)\mathbb{E}\left[p_{C_I}(T-t, Y_t^I, y)\right]dxdy; \\
&= \int \phi(x+y)p(T-t, X_t, x)\hat{p}_{C_I}(y)dxdy; \\
&= \int \phi(g)\left(\int p(T-t, X_t, x)\hat{p}_{C_I}(g-x)dx\right)dg; \\
&= \int \phi(g)u(t, X_t, g)dg.
\end{aligned}$$

Above the second equality holds by the Chapman-Kolmogorov equations. This shows that given $\mathcal{F}_t^B$, G_I has pdf $u(t, X_t, \cdot)$. As $\mathcal{F}_0^B$ is trivial the Jacod equivalence condition and first equality in (2.1) follow. The second equality in (2.1) follows from {(B.3)} and Itô's formula, finishing (1). As for (2), the right-continuity of $\mathbb{F}^m$ and that $1/\tilde{p}^{G_I}$ is a $(\mathbb{P}, \mathbb{F}^m)$ martingale follow from Lemma 4.3; while the second equality in (2.2) follows from Proposition 4.6. Lastly, the statement regarding B in part (3) follows from (Fontana, 2018, Proposition 2.9) the statement regarding B^m follows from (Fontana, 2018, Corollary 2.10). □

3. Lemmas Regarding the Partial Communication Equilibrium

We first prove lemmas in the Markovian noise setting. Assumptions {2.1}, {7.1}, {7.2}, and {A.1} are in force. Recall the signal H and market filtration $\mathbb{F}^m$ from Assumption {7.2}, and the function ℓ from {(4.5)} and {(7.2)}. The first lemma collects facts about $\mathbb{F}^m$ and the martingale preserving measure $\widetilde{\mathbb{P}}^H$ of {(7.4)}.

Lemma 3.1.

(1) *For each $t \leq T$, the law of H given $\mathcal{F}_t^B$ has pdf $\ell(t, X_t, \cdot)$. In particular, $\mathbb{P}\left[H \in \cdot \middle| \mathcal{F}_t^B\right] \sim \mathbb{P}\left[H \in \cdot\right]$ almost surely, with density*

$$p_t^h := \frac{\ell(t, X_t, h)}{\ell(0, X_0, h)} = \mathcal{E}\left(\int_0^{\cdot} (\mu_u^h)' dB_u\right)_t; \quad \mu_t^h := a(X_t)' \nabla_x \left(\log\left(\ell(t, X_t, h)\right)\right). \tag{3.1}$$

(2) *$\mathbb{F}^m$ is right-continuous, $1/p^H$ is a $(\mathbb{P}, \mathbb{F}^m)$-martingale, and the martingale preserving measure takes the form*

$$\frac{d\widetilde{\mathbb{P}}^H}{d\mathbb{P}} = \frac{1}{p_T^H}; \qquad p_{\cdot}^H = \mathcal{E}\left(\int_0^{\cdot} (\mu_u^H)' dB_u\right)_{\cdot}.$$

(3) *B is a $(\widetilde{\mathbb{P}}^H, \mathbb{F}^m)$ Brownian motion with the PRP, and $B_{\cdot}^m := B_{\cdot} - \int_0^{\cdot} \mu_u^H du$, is a $(\mathbb{P}, \mathbb{F}^m)$ Brownian motion with the PRP.*

Proof of Lemma 3.1. For (1), let $\phi \in C_c^\infty(\mathbb{R}^d)$ and $t \leq T$. By the tower property

$$\mathbb{E}\left[\phi(H) \middle| \mathcal{F}_t^B\right] = \mathbb{E}\left[\mathbb{E}\left[\phi(H(X_T + Y_T^I, \tau_N(X_T + Y_T^I) + Y_T^N)) \middle| \mathcal{F}_t^{B, W^I, W^N}\right] \middle| \mathcal{F}_t^B\right].$$

Using the Markov property

$$\begin{aligned}&\mathbb{E}\left[\phi(H(X_T + Y_T^I, \tau_N(X_T + Y_T^I) + Y_T^N)) \middle| \mathcal{F}_t^{B, W^I, W^N}\right] \\ &\quad = \int \phi(H(x + y, \tau_N(x + y) + \tilde{y})) p(T - t, X_t, x) p_I(t, Y_t^I, T, y) p_N(t, Y_t^N, T, \tilde{y}) dx dy d\tilde{y},\end{aligned}$$

where we integrate over $E \times \mathbb{R}^d \times \mathbb{R}^d$. The independence of (Y^I, Y^N, B), along with the Chapman-Kolmogorov equations imply

$$\mathbb{E}\left[p(T - t, X_t, x) p_I(t, Y_t^I, T, y) p_N(t, Y_t^N, T, \tilde{y}) \middle| \mathcal{F}_t^B\right] = p(T - t, X_t, x) \hat{p}_I(y) \hat{p}_N(\tilde{y}).$$

Therefore,

$$\mathbb{E}\left[\phi(H) \middle| \mathcal{F}_t^B\right] = \int \phi(H(x + y, \tau_N(x + y) + \tilde{y})) p(T - t, X_t, x) \hat{p}_I(y) \hat{p}_N(\tilde{y}) dx dy d\tilde{y}.$$

With $x, \tilde{y}$ fixed, letting $g = x + y$ gives

$$\int \phi(H(g, \tau_N g + \tilde{y})) p(T - t, X_t, x) \hat{p}_I(g - x) \hat{p}_N(\tilde{y}) dx dg d\tilde{y},$$

and we are integrating over $E \times \mathbb{R}^d \times \mathbb{R}^d$. Next, with x, g fixed, let $h = H(g, \tau_N g + \tilde{y})$ so $\tilde{y} = G(g, h) - \tau_N g$, $d\tilde{y} = |J^G|(g, h) dh$, and, by Assumption {7.2}, h takes values in

$\mathcal{R}_H$ for an integration region of $E\times\mathbb{R}^d\times\mathcal{R}_H$. This yields

$$\begin{aligned}&\int \phi(h)p(T-t,X_t,x)\hat{p}_I(g-x)\hat{p}_N(G(g,h)-\tau_N g)|J^G|(g,h)dxdgdh;\\ &\quad=\int\phi(h)\left(\int p(T-t,X_t,x)\hat{p}_I(g-x)\hat{p}_N(G(g,h)-\tau_N g)|J^G|(g,h)dxdg\right)dh;\\ &\quad=\int\phi(h)\ell(t,X_t,h)dh.\end{aligned}$$

Thus, given $\mathcal{F}^B_t$, H has pdf $\ell(t,X_t,\cdot)$. As $\mathcal{F}^B_0$ is trivial, the Jacod equivalence condition and first equality in (3.1) readily follow. The second equality in (3.1) holds by Itô's formula, since the PDE for u in {(B.3)}, implies that for a fixed h, ℓ solves $\ell_t+L\ell=0$ on $(0,T)\times E$. This finishes item (1). The statements in parts $(2)-(3)$ follow from the exact same argument used to prove parts $(2)-(3)$ in Lemma 2.1. □

We next prove similar results for $\mathbb{F}^I=\mathbb{F}^m\vee\mathfrak{s}(G_I)$ with G_I from Assumption {7.1}.

Lemma 3.2.

(1) For each $t\le T$, $\mathbb{P}\left[G_I\in\cdot\middle|\mathcal{F}^m_t\right]\sim\mathbb{P}\left[G_I\in\cdot\right]$ almost surely with density

$$p^{H,g}_t:=\frac{\hat{p}_N(G(g,H)-\tau_N g)|J^G|(g,H)}{\ell(t,X_t,H)}\times\frac{u(t,X_t,g)}{u(0,X_0,g)},\ g\in\mathbb{R}^d. \tag{3.2}$$

In particular, with $\tilde{p}^g,\tilde{\mu}^g$ from (2.1) and p^H,μ^H from (3.1) we have

$$\mathring{p}^{H,g}_t:=\frac{p^{H,g}_t}{p^{H,g}_0}=\frac{u(t,X_t,g)}{u(0,X_0,g)}\times\frac{\ell(0,X_0,H)}{\ell(t,X_t,H)}=\frac{\tilde{p}^g_t}{p^H_t}=\mathcal{E}\left(\int_0^\cdot(\tilde{\mu}^g_u-\mu^H_u)'dB^m_u\right)_t. \tag{3.3}$$

(2) $\mathbb{F}^I$ is right-continuous, $1/p^{H,G_I}$ is a $(\mathbb{P},\mathbb{F}^I)$ martingale, and the ($\mathbb{F}^m$ to $\mathbb{F}^I$) martingale preserving measure $\widetilde{\mathbb{P}}^{H,G_I}$ for G_I is defined by

$$\frac{d\widetilde{\mathbb{P}}^{H,G_I}}{d\mathbb{P}}:=\frac{1}{p^{H,G_I}_T};\quad p^{H,G_I}_\cdot=p^{H,G_I}_0\mathcal{E}\left(\int_0^\cdot(\mu^{H,G_I}_u)'dB^m_u\right)_\cdot;\quad \mu^{H,g}_\cdot:=\tilde{\mu}^g_\cdot-\mu^H_\cdot. \tag{3.4}$$

(3) B^m is a $(\widetilde{\mathbb{P}}^{m,G_I},\mathbb{F}^I)$ Brownian motion with the PRP, and $B^I_\cdot:=B^m_\cdot-\int_0^\cdot\mu^{H,G_I}_u du=B_\cdot-\int_0^\cdot\tilde{\mu}^{G_I}_u du$, is a $(\mathbb{P},\mathbb{F}^I)$ Brownian motion with the PRP.

Proof of Lemma 3.2. We start with part (1). As these calculations are similar to those in Lemmas 2.1, 3.1 we will typically omit explanations. Let $\phi,\psi\in C^\infty_c(\mathbb{R}^d)$ and $A_t\in\mathbb{F}^B_t$ for $t\le T$. First

$$\mathbb{E}\left[1_{A_t}\phi(G_I)\psi(H)\right]=\mathbb{E}\left[1_{A_t}\mathbb{E}\left[\phi(X_T+Y^I_T)\psi(H(X_T+Y^I_T,\tau_N(X_T+Y^I_T)+Y^N_T))\middle|\mathcal{F}^{B,W^I,W^N}_t\right]\right].$$

Next,

$$
\begin{aligned}
&\mathbb{E}\left[\phi(X_T+Y_T^I)\psi(H(X_T+Y_T^I,\tau_N(X_T+Y_T^I)+Y_T^N))\middle|\mathcal{F}_t^{B,W^I,W^N}\right]\\
&\quad=\int\phi(x+y)\psi(H(x+y,\tau_N(x+y)+\tilde{y}))p(T-t,X_t,x)\hat{p}_I(y)\hat{p}_N(\tilde{y})dxdyd\tilde{y};\\
&\quad=\int\phi(g)\psi(H(g,\tau_Ng+\tilde{y}))p(T-t,X_t,x)\hat{p}_I(g-x)\hat{p}_N(\tilde{y})dxdgd\tilde{y};\\
&\quad=\int\phi(g)\psi(H(g,\tau_Ng+\tilde{y}))u(t,X_t,g)\hat{p}_N(\tilde{y})dgd\tilde{y}.
\end{aligned}
$$

With g fixed, set $h=H(g,\tau_Ng+\tilde{y})$ so that $\tilde{y}=G(g,h)-\tau_Ng$, $d\tilde{y}=|J^G|(g,h)dh$. Additionally multiplying by 1_{A_t} and taking expectations yields

(3.5)
$$
\mathbb{E}\left[1_{A_t}\phi(G_I)\psi(H)\right]=\mathbb{E}\left[1_{A_t}\int\phi(g)\psi(h)\hat{p}_N(G(g,h)-\tau_Ng)|J^G|(g,h)u(t,X_t,g)dgdh\right].
$$

By Lemma 3.1 we know that given $\mathcal{F}_t^B$, H has pdf $\ell(t,X_t,\cdot)$. Therefore, for any suitably measurable and integrable function χ

$$
\mathbb{E}\left[1_{A_t}\chi(t,X_t,H)\psi(H)\right]=\mathbb{E}\left[1_{A_t}\int\psi(h)\chi(t,X_t,h)\ell(t,X_t,h)dh\right].
$$

Thus, with

$$
\chi(t,x,h)=\frac{1}{\ell(t,x,h)}\times\int\phi(g)\hat{p}_N(G(g,h)-\tau_Ng)|J^G|(g,h)u(t,x,g)dg,
$$

we see that $\mathbb{E}\left[1_{A_t}\phi(G_I)\psi(H)\right]=\mathbb{E}\left[1_{A_t}\chi(t,X_t,H)\psi(H)\right]$ for all A_t,ψ and hence

$$
\mathbb{E}\left[\phi(G_I)\middle|\mathcal{F}_t^m\right]=\chi(t,X_t,H)=\int\phi(g)\frac{\hat{p}_N(G(g,H)-\tau_Ng)|J^G|(g,H)u(t,X_t,g)}{\ell(t,X_t,H)}dg,
$$

so that given $\mathcal{F}_t^m$, G_I has pdf

$$
\frac{\hat{p}_N(G(g,H)-\tau_Ng)|J^G|(g,H)u(t,X_t,g)}{\ell(t,X_t,H)}.
$$

This shows $p_t^{H,g}$ is the density of $\mathbb{P}\left[G_I\in\cdot\middle|\mathcal{F}_t^m\right]$ with respect to $\mathbb{P}\left[G_I\in\cdot\right]$ since G_I has unconditional pdf $u(0,X_0,\cdot)$. The statement in (3.3) follows from (2.1), (3.1) and part (3) of Lemma 3.1. The statements in (2) follow from Lemma 4.3 and Proposition 4.6. The statements in (3) follow from (Fontana, 2018, Proposition 2.9, Corollary 2.10). □

Continuing, we prove results about the filtration $\mathbb{F}^N=\mathbb{F}^m\vee\mathfrak{s}(G_N)$ for G_N from Assumption {7.2}. To state the Lemma, recall the pdf for G_N given in {(C.1)}.

Lemma 3.3.

(1) For each $t \leq T$, $\mathbb{P}\left[G_N \in \cdot \middle| \mathcal{F}_t^m\right] \sim \mathbb{P}\left[G_N \in \cdot\right]$ almost surely with density

$$p_{N,t}^{H,g} := \frac{u(t, X_t, \check{G}(g,H))|J^{\check{G}}|(g,H)\hat{p}_N(g - \tau_N \check{G}(g,H))}{\ell(t, X_t, H)u_N(g)}. \tag{3.6}$$

In particular, with $p^{H,g}$ from (3.2)

$$\frac{p_t^{H,G_I}}{p_{N,t}^{H,G_N}} = \frac{p_0^{G_I,H}}{p_{N,0}^{G_N,H}} = \frac{|J^G|(G_I,H)u_N(G_N)}{|J^{\check{G}}|(G_N,H)u(0,X_0,G_I)}. \tag{3.7}$$

(2) $\mathbb{F}^N = \mathbb{F}^I$

Proof of Lemma 3.3. We start with (1). These calculations are very similar to those in Lemma 3.2, and as such, we will not include all the steps. Let $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$ and $A_t \in \mathbb{F}_t^B$ for $t \leq T$. First

$$\begin{aligned}
&\mathbb{E}\left[\phi(G_N)\psi(H)\middle|\mathcal{F}_t^{B,W^I,W^N}\right] \\
&\quad = \mathbb{E}\left[\phi(\tau_N(X_T + Y_T^I) + Y_T^N)\psi(H(X_T + Y_T^I, \tau_N(X_T + Y_T^I) + Y_T^N))\middle|\mathcal{F}_t^{B,W^I,W^N}\right] \\
&\quad = \int \phi(\tau_N g + \tilde{y})\psi(H(g, \tau_N g + \tilde{y}))p(T-t, X_t, x)\hat{p}_I(g - x)\hat{p}_N(\tilde{y})dxdgd\tilde{y}; \\
&\quad = \int \phi(\tau_N g + \tilde{y})\psi(H(g, \tau_N g + \tilde{y}))u(t, X_t, g)\hat{p}_N(\tilde{y})dgd\tilde{y}; \\
&\quad = \int \phi(z)\psi(H(g,z))u(t, X_t, g)\hat{p}_N(z - \tau_N g)dgdz.
\end{aligned}$$

For z fixed, set $h = H(g,z)$ so that $g = \check{G}(z,h)$, $dg = |J^{\check{G}}|(z,h)dh$. This leads to

$$\int \phi(z)\psi(h)u(t, X_t, \check{G}(z,h))|J^{\check{G}}|(z,h)\hat{p}_N(z - \tau_N \check{G}(z,h))dzdh.$$

Therefore,

$$\mathbb{E}\left[1_{A_t}\phi(G_N)\psi(H)\right] = \mathbb{E}\left[1_{A_t}\int \phi(z)\psi(h)u(t, X_t, \check{G}(z,h))|J^{\check{G}}|(z,h)\hat{p}_N(z - \tau_N \check{G}(z,h))dzdh\right].$$

Repeating the analogous steps as in Lemma 3.2 we deduce that

$$\mathbb{E}\left[\phi(G_N)\middle|\mathcal{F}_t^m\right] = \int \phi(z)\frac{u(t, X_t, \check{G}(z,H))|J^{\check{G}}|(z,H)\hat{p}_N(z - \tau_N \check{G}(z,H))}{\ell(t, X_t, H)}dz.$$

so that given F_t^m, G_N has pdf (replacing g with z)

$$\frac{u(t, X_t, \check{G}(g,H))|J^{\check{G}}|(g,H)\hat{p}_N(g - \tau_N \check{G}(g,H))}{\ell(t, X_t, H)}$$

and hence (3.6) follows, as u_N is the pdf for G_N. The identity in (3.7) is immediate. Lastly, $\mathbb{F}^N = \mathbb{F}^I$ follows because $H(x,y)$ is invertible in both x and y. $\square$

With all the preparatory lemmas in place, we prove Theorem {7.3}. Note that by Assumption {7.1}, $\hat{p}_I$, and hence u, is bounded from above. Similarly, Assumptions {7.1}, {7.2} and Lemma 3.1 imply ℓ is bounded from above, and thus we deduce

$$(3.8)\qquad \begin{aligned} &\int_{\mathbb{R}^d} \left(\log\left(u(0,X_0,g)\right)\right)^+ \left(u(0,X_0,g)+u_N(g)\right)dg < \infty;\\ &\int_{\mathbb{R}^d\times\mathcal{R}_H} \left(\log\left(u(0,X_0,G(g,h)\right)\right)^+ u(0,X_0,g)\ell(0,X_0,h)dgdh < \infty;\\ &\int_{\mathcal{R}_H} \left(\log\left(\ell(0,X_0,h)\right)\right)^+ \ell(0,X_0,h)dh < \infty. \end{aligned}$$

Proof of Theorem {7.3}. We start collecting facts regarding $\check{\mathbb{Q}}$. First, $B^{\check{\mathbb{Q}}}_\cdot := B_\cdot + \int_0^\cdot \check{\nu}_u du$ is a $(\check{\mathbb{Q}},\mathbb{F}^m)$ Brownian motion with the PRP. Next, S is a $(\check{\mathbb{Q}},\mathbb{F}^m)$ martingale by construction, with $dS_t = \sigma_t dB^{\check{\mathbb{Q}}}_t = \sigma_t(\check{\nu}_t + dB_t)$. Lastly,

$$(3.9)\qquad Z^{\check{\mathbb{Q}}}_t = \frac{d\check{\mathbb{Q}}}{d\widetilde{\mathbb{P}}}\Big|_{\mathcal{F}^m_t} \times \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}^m_t} = \frac{\check{Z}_t}{p^H_t} = \frac{\check{Z}_t \ell(0,X_0,H)}{\ell(t,X_t,H)} = \mathcal{E}\left(-\int_0^\cdot \nu'_u dB^m_u\right)_t$$

for $t\leq T$. Indeed, the second equality follows from {(7.4)}, {(C.2)}; the third equality from (3.1); and the fourth equality from (3.1), {(C.2)} and $dB^m_t = dB_t - \mu^h_t dt$. Now, consider the uninformed investor's value function. Clearly, $\check{\mathbb{Q}}\in\tilde{\mathcal{M}}$ and in fact, we claim $\check{\mathbb{Q}}\in\tilde{\mathcal{M}}^m$. First, $Z^{\check{\mathbb{Q}}}_0 = 1$ so $Z^{\check{\mathbb{Q}}} = \mathring{Z}^{\check{\mathbb{Q}}}$. Second, as $\ell(0,X_0,H)$ is $\mathcal{F}^m_0$ measurable; $\widetilde{\mathbb{E}}\left[\check{Z}_T\middle|\mathcal{F}^m_0\right] = 1$; $\widetilde{\mathbb{P}} = \mathbb{P}$ on $\sigma(H)$; and $H\sim \ell(0,X_0,\cdot)$

$$(3.10)\qquad \begin{aligned} \mathbb{E}\left[\mathring{Z}^{\check{\mathbb{Q}}}_T \log\left(\mathring{Z}^{\check{\mathbb{Q}}}_T\right)\right] \leq \widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\check{Z}_T\right)\right] + \int_{h\in\mathcal{R}_H} \left(\log\left(\ell(0,X_0,h)\right)\right)^+ \ell(0,X_0,h)dh\\ + \widetilde{\mathbb{E}}\left[\check{Z}_T \log\left(\ell(T,X_T,H)\right)^-\right] < \infty, \end{aligned}$$

where the last inequality holds from (3.8), {(C.3)}, and {(C.4)}(a). Thus, $H_0\left(\check{\mathbb{Q}}\middle|\mathbb{P}\right) < \infty$ almost surely so $\check{\mathbb{Q}}\in\tilde{\mathcal{M}}^m$. Next, we claim for all $\mathbb{Q}\in\mathcal{M}^m$

$$(3.11)\qquad Z^{\mathbb{Q}}_\cdot = Z^{\mathbb{Q}}_0 \mathring{Z}^{\mathbb{Q}}_\cdot,$$

so $\mathring{Z}^{\mathbb{Q}} = \mathring{Z}^{\check{\mathbb{Q}}}$. Indeed, from Lemma 3.1 part (3), we can write $Z^{\mathbb{Q}}_T = Z^{\mathbb{Q}}_0 \mathcal{E}\left(\int_0^\cdot \theta'_t dB^m_t\right)_T$ for some $\theta\in\mathcal{P}(\mathbb{F}^m)$. Girsanov's theorem and {(7.6)} imply S has dynamics

$$dS_t = \sigma_t\left((\nu_t+\theta_t)dt + dB^{\mathbb{Q}}_t\right);\quad B^{\mathbb{Q}}_\cdot = B^m_\cdot - \int_0^\cdot \theta_u du.$$

$\mathbb{Q} \in \mathcal{M}^m$ implies $\int_0^\cdot \sigma_u(\nu_u + \theta_u)du$ is a continuous $(\mathbb{Q}, \mathbb{F}^m)$-local martingale of finite variation, and hence identically zero. This gives that $\mathsf{Leb}_{[0,T]} \times \mathbb{P}$ almost surely that $\theta = -\nu$, which in light of (3.9) verifies (3.11).

By duality, for each $\pi \in \mathcal{A}^m$, $\mathbb{Q} \in \tilde{\mathcal{M}}^m$, and $\mathcal{F}_0^m$ measurable $\lambda > 0$

$$\begin{aligned}\mathbb{E}\left[-\frac{1}{\gamma_U}e^{-\gamma_U \mathcal{W}_T^\pi}\big|\mathcal{F}_0^m\right] &\leq \mathbb{E}\left[\frac{1}{\gamma_U}(\lambda Z_T^{\mathbb{Q}})\left(\log(\lambda Z_T^{\mathbb{Q}}) - 1\right) + \lambda Z_T^{\mathbb{Q}}\mathcal{W}_T^\pi\big|\mathbb{F}_0^m\right] \\ &\leq \frac{1}{\gamma_U}\mathring{\lambda}(\log(\mathring{\lambda} - 1) + \frac{1}{\gamma_U}\mathring{\lambda}\mathbb{E}\left[\frac{\check{Z}_T}{p_T^H}\log\left(\frac{\check{Z}_T}{p_T^H}\right)\big|\mathcal{F}_0^m\right],\end{aligned}$$

where we have set $\mathring{\lambda} = \lambda Z_0^{\mathbb{Q}}$ and used (3.11). The infimum above over $\mathring{\lambda}$ is achieved at $\log(\mathring{\lambda}) = -\mathbb{E}\left[(\check{Z}_T/p_T^H)\log\left(\check{Z}_T/p_T^H\right)\big|\mathcal{F}_0^m\right]$, and plugging this in yields

$$\mathbb{E}\left[-\frac{1}{\gamma_U}e^{-\gamma \mathcal{W}_T^\pi}\big|\mathcal{F}_0^m\right] \leq -\frac{1}{\gamma_U}e^{-\mathbb{E}\left[\frac{\check{Z}_T}{p_T^H}\log\left(\frac{\check{Z}_T}{p_T^H}\right)\big|\mathcal{F}_0^m\right]}.$$

Furthermore, there is equality if and only if

$$\mathcal{W}_T^\pi = \frac{1}{\gamma_U}\mathbb{E}\left[\frac{\check{Z}_T}{p_T^H}\log\left(\frac{\check{Z}_T}{p_T^H}\right)\big|\mathcal{F}_0^m\right] - \frac{1}{\gamma_U}\log\left(\frac{\check{Z}_T}{p_T^H}\right), \tag{3.12}$$

and $\mathbb{E}\left[(\check{Z}_T/p_T^H)\mathcal{W}_T^\pi\big|\mathcal{F}_0^m\right] = 0$, but this latter equality is immediate. By (3.9), (3.10) and $|x\log(x)| \leq x\log(x) + 2/e, x > 0$ we know

$$M_\cdot^U := -\frac{1}{\gamma_U}\mathbb{E}^{\check{\mathbb{Q}}}\left[\log\left(\frac{\check{Z}_T}{p_T^H}\right)\big|\mathcal{F}_\cdot^m\right],$$

is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$-martingale. Thus, by PRP there is a $\theta^U \in \mathcal{P}(\mathbb{F}^m)$ such that $\int_0^T |\theta_u^U|^2 du < \infty$, and $M_\cdot^U = M_0^U + \int_0^\cdot (\theta_u^U)' dB_u^{\check{\mathbb{Q}}}$. As σ is invertible and $dS_t = \sigma_t dB_t^{\check{\mathbb{Q}}}$, if we set $\hat{\pi}_\cdot^U := (\sigma_\cdot')^{-1}\theta_\cdot^U$, then $\mathcal{W}_\cdot^{\hat{\pi}^U} = M_\cdot^U - M_0^U$ is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ martingale verifying (3.12). Using (3.1), (3.9), noting that $\ell(0, X_0, H)$ is $\mathcal{F}_0^m$ measurable, and $\widetilde{\mathbb{E}}\left[\check{Z}_T\big|\mathcal{F}_0^m\right] = 1$, we simplify (3.12) to deduce the existence of an optimal strategy $\hat{\pi}^U \in \mathcal{A}^m$ such that

$$\begin{aligned}&\mathcal{W}_T^{\hat{\pi}^U} = \frac{1}{\gamma_U}\widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\frac{\check{Z}_T}{\ell(T, X_T, H)}\right)\big|\mathcal{F}_0^m\right] - \frac{1}{\gamma_U}\log\left(\frac{\check{Z}_T}{\ell(T, X_T, H)}\right); \\ &\mathcal{W}^{\hat{\pi}^U} \text{ is a } (\check{\mathbb{Q}}, \mathbb{F}^m) \text{ Martingale.}\end{aligned} \tag{3.13}$$

We next consider the noise trader's value function for a fixed $g \in \mathbb{R}^d$. Since $\mathring{p}^{H,g}$ from (3.3) is a strictly positive $(\mathbb{P}, \mathbb{F}^m)$ martingale (c.f. (3.3) and Lemma 4.3), the identity in (3.3) implies $\mathbb{P}^g$ from $\{(2.8)\}$ is well defined. We have already shown

$\check{\mathbb{Q}} \in \mathcal{M}$. Furthermore from {(B.3)}, (3.3) and (3.9) we know

$$Z_T^{\check{\mathbb{Q}},g} := \frac{d\check{\mathbb{Q}}}{d\mathbb{P}^g}\Big|_{\mathcal{F}_T^m} = \frac{d\check{\mathbb{Q}}}{d\mathbb{P}}\Big|_{\mathcal{F}_T^m} \times \frac{d\mathbb{P}}{d\mathbb{P}^g}\Big|_{\mathcal{F}_T^m} = \frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,g}} = \frac{\check{Z}_T u(0, X_0, g)}{\hat{p}_I(g - X_T)}. \tag{3.14}$$

As $Z_0^{\check{\mathbb{Q}},g} = 1$,

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}^g}\left[\mathring{Z}_T^{\check{\mathbb{Q}},g} \log\left(\mathring{Z}_T^{\check{\mathbb{Q}},g}\right)\right] = \mathbb{E}^{\check{\mathbb{Q}}}\left[\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,g}}\right)\right] = \widetilde{\mathbb{E}}\left[\check{Z}_T \log\left(\frac{\check{Z}_T u(0, X_0, g)}{\hat{p}_I(g - X_T)}\right)\right] \\ &\quad \leq \widetilde{\mathbb{E}}\left[\check{Z}_T \log\left(\check{Z}_T\right)\right] + \left(\log\left(u(0, X_0, g)\right)\right)^+ + \widetilde{\mathbb{E}}\left[\check{Z}_T \left(\log\left(\hat{p}_I(g - X_T)\right)\right)^-\right] < \infty. \end{aligned} \tag{3.15}$$

Above, the finiteness follows from {(C.3)} and {(C.4)}(b). Thus $H_0\left(\check{\mathbb{Q}}\middle|\mathbb{P}^g\right) < \infty$ and $\tilde{\mathcal{M}}^{m,g} \neq \emptyset$. Now, let $\mathbb{Q} \in \tilde{\mathcal{M}}^{m,g}$. From (3.11) we know $Z_\cdot^{\mathbb{Q}} = Z_0^{\mathbb{Q}} \check{Z}_\cdot / p_\cdot^H$. Thus, with respect to $\mathbb{P}^g$, $Z_\cdot^{\mathbb{Q},g} = Z_0^{\mathbb{Q}} \check{Z}_\cdot / (p_\cdot^H \mathring{p}_\cdot^{H,g})$. Therefore, repeating the duality argument, a strategy $\pi^{N,g}$ is optimal if and only if

$$\mathcal{W}_T^{\pi^{N,g}} = \frac{1}{\gamma_N} \mathbb{E}^{\mathbb{P}^g}\left[\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,g}} \log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,g}}\right) \big| \mathcal{F}_0^m\right] - \frac{1}{\gamma_N} \log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,g}}\right). \tag{3.16}$$

By (3.15) and $|x\log(x)| \leq x\log(x) + 2/e, x > 0$ we know

$$M_\cdot^{N,g} := -\frac{1}{\gamma_N} \mathbb{E}^{\check{\mathbb{Q}}}\left[\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,g}}\right) \big| \mathcal{F}_\cdot^m\right],$$

is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ martingale. Thus, there is $\theta^{N,g} \in \mathcal{P}(\mathbb{F}^m)$ so that $M_\cdot^{N,g} = M_0^{N,g} + \int_0^\cdot (\theta_u^{N,g})' dB_u^{\check{\mathbb{Q}}}$. If we set

$$\hat{\pi}_\cdot^{N,g} := (\sigma_\cdot')^{-1} \theta_\cdot^{N,g}, \tag{3.17}$$

then $\mathcal{W}_\cdot^{\hat{\pi}^{N,g}} = M^{N,g} \cdot - M_0^{N,g}$ is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ martingale verifying (3.16). Using (3.14), (3.15) this simplifies to

$$\mathcal{W}_T^{\hat{\pi}^{N,g}} = \frac{1}{\gamma_N} \widetilde{\mathbb{E}}\left[\check{Z}_T \log\left(\frac{\check{Z}_T}{\hat{p}_I(g - X_T)}\right) \big| \mathcal{F}_0^m\right] - \frac{1}{\gamma_N} \log\left(\frac{\check{Z}_T}{\hat{p}_I(g - X_T)}\right),$$

where the last equality follows as $u(0, X_, g)$ is constant and $\widetilde{\mathbb{E}}\left[\check{Z}_T \middle| \mathcal{F}_0^m\right] = 1$. Thus, we have shown the existence of an optimal strategy $\hat{\pi}^{N,g} \in \mathcal{A}^{N,g}$ which satisfies

$$\begin{aligned} &\mathcal{W}_T^{\hat{\pi}^{N,g}} = \frac{1}{\gamma_N} \widetilde{\mathbb{E}}\left[\check{Z}_T \log\left(\frac{\check{Z}_T}{\hat{p}_I(g - X_T)}\right) \big| \mathcal{F}_0^m\right] - \frac{1}{\gamma_N} \log\left(\frac{\check{Z}_T}{\hat{p}_I(g - X_T)}\right); \\ &\mathcal{W}^{\hat{\pi}^{N,g}} \text{ is a } (\check{\mathbb{Q}}, \mathbb{F}^m) \text{ Martingale.} \end{aligned} \tag{3.18}$$

We next turn to the informed investor. (3.9) implies $\check{Z}/p^H$ is a $(\mathbb{P}, \mathbb{F}^m)$ martingale starting at 1. Thus, if we define $\check{\mathbb{Q}}^I$ through $d\check{\mathbb{Q}}^I/d\mathbb{P} = \check{Z}_T / (p_T^H p_T^{H,G_I})$, then Lemma

4.8, part (2) implies $\check{\mathbb{Q}}^I \in \mathcal{M}^I$, while calculation shows $\mathring{Z}^{\check{\mathbb{Q}}^I} = \check{Z}/(p^H \mathring{p}^{H,G_I})$. Furthermore, $B^{\check{\mathbb{Q}}}$ is a $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$-Brownian motion with the PRP (c.f. (Fontana, 2018, Proposition 2.9) and Remark 4.4). Now, $\check{\mathbb{Q}}^I \in \tilde{\mathcal{M}}^I$ will follow if $\mathbb{E}\left[\mathring{Z}_T^{\check{\mathbb{Q}}^I} \log(\mathring{Z}_T^{\check{\mathbb{Q}}^I})\big|\mathcal{F}_0^I\right] < \infty$. To show this, we first claim (note the presence of $\mathring{Z}^{\check{\mathbb{Q}}^I}$):

$$\mathbb{E}^{\check{\mathbb{Q}}^I}\left[\left|\log\left(\mathring{Z}_T^{\check{\mathbb{Q}}^I}\right)\right|\right] = \mathbb{E}\left[\frac{\check{Z}_T}{p_T^H p_T^{H,G_I}}\left|\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}}\right)\right|\right] < \infty. \tag{3.19}$$

Since $(x/y)|\log(x)| \leq (1/y)\left(x\log(x) + 2/e\right)$ for $x, y > 0$, we see (for $x = \check{Z}_T/(p_T^H \mathring{p}_T^{H,G_I})$ and $y = p_0^{H,G_I}$)

$$\mathbb{E}\left[\frac{\check{Z}_T}{p_T^H p_T^{H,G_I}}\left|\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}}\right)\right|\right] \leq \mathbb{E}\left[\frac{\check{Z}_T}{p_T^H p_T^{H,G_I}}\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}}\right) + \frac{2}{e p_0^{H,G_I}}\right]. \tag{3.20}$$

By definition of p^{H,G_I}, and since $G_I \sim u(0, X_0, \cdot)$

$$\mathbb{E}\left[\frac{1}{p_0^{H,G_I}}\right] = \mathbb{E}\left[\int_{g\in\mathbb{R}^d} \frac{1}{p_0^{H,g}} p_0^{H,g} u(0, X_0, g)dg\right] = 1. \tag{3.21}$$

Next,

$$\begin{aligned}
\mathbb{E}\left[\frac{\check{Z}_T}{p_T^H p_T^{H,G_I}}\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}}\right)\right] &= \mathbb{E}\left[\frac{\check{Z}_T}{p_T^H}\int_{\mathbb{R}^d}\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,g}}\right)u(0, X_0, g)dg\right] \\
&= \int_{\mathbb{R}^d}\widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\frac{\check{Z}_T u(0, X_0, g)}{\hat{p}_I(g - X_T)}\right)\right]u(0, X_0, g)dg \\
&\leq \widetilde{\mathbb{E}}\left[\check{Z}_T\log(\check{Z}_T)\right] + \int\limits_{g\in\mathbb{R}^d}\widetilde{\mathbb{E}}\left[\check{Z}_T\left(\log\left(\hat{p}_I(g - X_T)\right)\right)^-\right]u(0, X_0, g)dg \\
&\quad + \int\limits_{g\in\mathbb{R}^d}\left(\log\left(u(0, X_0, g)\right)\right)^+ u(0, X_0, g)dg \quad < \infty.
\end{aligned} \tag{3.22}$$

Above, the first equality follows by conditioning on $\mathcal{F}_T^m$ and using part (1) of Lemma 3.2; the second equality using {(B.3)}, (3.1) and (3.3); and the second inequality follows from {(C.3)}, (3.8) and {(C.4)}(c). Therefore, (3.19) follows from (3.20), (3.21), (3.22). But, (3.19) implies $\mathbb{E}\left[\mathring{Z}_T^{\check{\mathbb{Q}}^I}\log(\mathring{Z}_T^{\check{\mathbb{Q}}^I})\big|\mathcal{F}_0^I\right] < \infty$ since

$$0 \leq \mathbb{E}\left[\mathring{Z}_T^{\check{\mathbb{Q}}^I}\log\left(\mathring{Z}_T^{\check{\mathbb{Q}}^I}\right)\big|\mathcal{F}_0^I\right] \leq \frac{1}{Z_0^{\check{\mathbb{Q}}^I}}\mathbb{E}\left[Z_T^{\check{\mathbb{Q}}^I}\left|\log\left(\mathring{Z}_T^{\check{\mathbb{Q}}^I}\right)\right|\big|\mathcal{F}_0^I\right].$$

Next, we claim any $\mathbb{Q} \in \tilde{\mathcal{M}}^I$ has density process $Z_\cdot^{\mathbb{Q}} = Z_0^{\mathbb{Q}}\mathring{Z}_\cdot^{\check{\mathbb{Q}}^I}$. Indeed, using part (4) of Lemma 3.2, we deduce the existence of $\theta \in \mathcal{P}(\mathbb{F}^I)$ so that $Z_T^{\mathbb{Q}} = Z_0^{\mathbb{Q}}\mathcal{E}\left(\int_0^\cdot \theta_t' dB_t^I\right)_T$. Using Girsanov, along with $dB_t^I = dB_t^m - \mu_t^{H,G_I}dt$ and {(7.6)}, we see S has dynamics

$dS_t = \sigma_t(\nu_t + \mu_t^{H,G_I} + \theta_t)dt + \sigma_t dB_t^{\mathbb{Q},I}$ where $B_\cdot^{\mathbb{Q},I} = B_\cdot^I - \int_0^\cdot \theta_u du$. As S, $B^{\mathbb{Q},I}$ are continuous $(\mathbb{Q}, \mathbb{F}^I)$-local martingales, $\nu + \mu^{H,G_I} + \theta \equiv 0$ from whence

$$Z_t^{\mathbb{Q}} = Z_0^{\mathbb{Q}} \mathcal{E}\left(-\int_0^\cdot (\nu_u + \mu_u^{H,G_I})' dB_u^I\right)_t = Z_0^{\mathbb{Q}} \frac{\check{Z}_t}{p_t^H \mathring{p}_t^{H,G_I}}.$$

To obtain the last equality we use the following steps:

$$\begin{aligned}\frac{\check{Z}_t}{p_t^H \mathring{p}_t^{H,G_I}} &= \frac{\check{Z}_t u(0, X_0, G_I)}{u(t, X_t, G_I)} = \frac{\mathcal{E}\left(-\int_0^\cdot \check{\nu}_u' dB_u\right)_t}{\mathcal{E}\left(\int_0^\cdot (\tilde{\mu}_u^{G_I})' dB_u\right)_t} = \mathcal{E}\left(-\int_0^\cdot (\check{\nu}_u + \tilde{\mu}_u^{G_I})'(dB_u - \tilde{\mu}_u^{G_I} du)\right)_t \\ &= \mathcal{E}\left(-\int_0^\cdot (\nu_u + \mu_u^{H,G_I})'(dB_u^m - \mu_u^{H,G_I} du)\right)_t = \mathcal{E}\left(-\int_0^\cdot (\nu_u + \mu_u^{H,G_I})' dB_u^I\right)_t\end{aligned}$$

The first equality follows from (3.3); the second from (2.2), part (3) of Lemma 3.2; the fourth from $\nu = \check{\nu} + \mu^H$, $dB_t^m = dB_t - \mu_t^H dt$, (3.4); and the fifth also from part (3) of Lemma 3.2. This proves the assertion.

The duality argument shows π^I is optimal if and only if

$$\mathcal{W}_T^{\pi^I} = \frac{1}{\gamma_I} \mathbb{E}\left[\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}} \log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}}\right) \big| \mathcal{F}_0^I\right] - \frac{1}{\gamma_I} \log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}}\right).$$

By (3.19) we know

$$M_\cdot^I := -\frac{1}{\gamma_I} \mathbb{E}^{\check{\mathbb{Q}}^I}\left[\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}}\right) \big| \mathcal{F}_\cdot^I\right],$$

is a $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$-martingale. Thus, there is a $\theta^I \in \mathcal{P}(\mathbb{F}^I)$ such that $M_t^I = M_0^I + \int_0^t (\theta_u^I)' dB_u^{\check{\mathbb{Q}}}$. Thus, if we set $\hat{\pi}_\cdot^I := (\sigma_\cdot')^{-1} \theta_\cdot^I$, then $\mathcal{W}_\cdot^{\hat{\pi}^I} = M_\cdot^I - M_0^I$ is a $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$-martingale, and using the conditional Bayes rule

$$\begin{aligned}\mathcal{W}_T^{\hat{\pi}^I} &= \frac{1}{\gamma_I} \mathbb{E}^{\check{\mathbb{Q}}^I}\left[\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}}\right) \big| \mathcal{F}_0^I\right] - \frac{1}{\gamma_I} \log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}}\right) \\ &= \frac{1}{\gamma_I} \mathbb{E}\left[\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}} \log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}}\right) \big| \mathcal{F}_0^I\right] - \frac{1}{\gamma_I} \log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_I}}\right),\end{aligned}$$

proving optimality of $\hat{\pi}^I$. To simplify this expression note that as above $p_T^H \mathring{p}_T^{H,G_I} = \hat{p}_I(G_I - X_T)/u(0, X_0, G_I)$, and since $u(0, X_0, G_I)$ is $\mathcal{F}_0^I$ measurable it disappears from the expression for $\mathcal{W}_T^{\hat{\pi}^I}$. Furthermore, Lemma 4.8 implies

$$\begin{aligned}\mathbb{E}^{\check{\mathbb{Q}}^I}\left[\log\left(\frac{\check{Z}_T}{\hat{p}_I(G_I - X_T)}\right)\big|\mathcal{F}_0^I\right] &= \left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\frac{\check{Z}_T}{\hat{p}_I(g - X_T)}\right)\big|\mathcal{F}_0^m\right]\right)\big|_{g=G_I}\\ &= \widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\check{Z}_T\right)\big|\mathcal{F}_0^m\right]\\ &\quad - \left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\hat{p}_I(g - X_T)\right)\big|\mathcal{F}_0^m\right]\right)\big|_{g=G_I}.\end{aligned}$$

We conclude for the optimal trading strategy $\hat{\pi}^I$

$$\begin{aligned}\mathcal{W}_T^{\hat{\pi}^I} &= \frac{1}{\gamma_I}\left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\check{Z}_T\right)\big|\mathcal{F}_0^m\right] - \left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\hat{p}_I(g - X_T)\right)\big|\mathcal{F}_0^m\right]\right)\big|_{g=G_I}\right)\\ &\quad - \frac{1}{\gamma_I}\log\left(\frac{\check{Z}_T}{\hat{p}_I(G_I - X_T)}\right);\\ \mathcal{W}^{\hat{\pi}^I} &\text{ is a } (\check{\mathbb{Q}}^I, \mathbb{F}^I) \text{ Martingale.}\end{aligned} \tag{3.23}$$

Having identified the optimal wealth processes for each investor, we now put them together. We have already shown $\mathcal{W}^{\hat{\pi}^I}$ is a $(\check{\mathbb{Q}}, \mathbb{F}^I)$-martingale, and from (Amendinger, 2000, Proposition 3.4), $\mathbb{F}^m \subseteq \mathbb{F}^I$, we know $\hat{\pi}^U$ is both $\mathbb{F}^I$ predictable and S integrable under $\check{\mathbb{Q}}^I$. Furthermore, the semi-martingales $\mathbb{F}^m - \mathcal{W}^{\hat{\pi}^U}, \mathbb{F}^I - \mathcal{W}^{\hat{\pi}^U}$ have a common version. Thus, as (3.13) implies $\mathcal{W}^{\hat{\pi}^U}$ is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ martingale, (Amendinger, 2000, Theorem 3.2) shows $\mathcal{W}^{\hat{\pi}^U}$ is a $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$ martingale. As for the noise trader, Lemma 3.4 below proves

$$\begin{aligned}&\hat{\pi}^{N,G_N} \text{ is well-defined, } \mathbb{F}^I \text{ predictable and } S \text{ integrable under } (\check{\mathbb{Q}}^I, \mathbb{F}^I).\\ &\mathcal{W}^{\hat{\pi}^{N,G_N}} \text{ is a } (\check{\mathbb{Q}}^I, \mathbb{F}^I) \text{ martingale and } \mathcal{W}^{\hat{\pi}^{N,G_N}} = \left(\mathcal{W}^{\hat{\pi}^{N,g}}\right)\big|_{g=G_N}.\end{aligned} \tag{3.24}$$

Therefore, we use (3.13), (3.23), (3.18) to obtain

$$\begin{aligned}&\int_0^T\left(\omega_U\hat{\pi}_t^U + \omega_I\hat{\pi}_t^I + \omega_N\hat{\pi}_t^{N,G_N}\right)' dS_t = \omega_U\mathcal{W}_T^{\hat{\pi}^U} + \omega_I\mathcal{W}_T^{\hat{\pi}^I} + \omega_N\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\\ &= \alpha_U\left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log(\check{Z}_T)\big|\mathcal{F}_0^m\right] - \widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\ell(T, X_T, H)\right)\big|\mathcal{F}_0^m\right] - \log\left(\frac{\check{Z}_T}{\ell(T, X_T, H)}\right)\right)\\ &+ \alpha_I\left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log(\check{Z}_T)\big|\mathcal{F}_0^m\right] - \left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log(\hat{p}_I(g - X_T))\big|\mathcal{F}_0^m\right]\right)_{g=G_I} - \log\left(\frac{\check{Z}_T}{\hat{p}_I(G_I - X_T)}\right)\right)\\ &+ \alpha_N\left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log(\check{Z}_T)\big|\mathcal{F}_0^m\right] - \left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log(\hat{p}_I(g - X_T))\big|\mathcal{F}_0^m\right]\right)_{g=G_N} - \log\left(\frac{\check{Z}_T}{\hat{p}_I(G_N - X_T)}\right)\right).\end{aligned}$$

Furthermore,

$$\overline{\mathbf{M}} := \int_0^{\cdot}\left(\omega_U\hat{\pi}_t^U + \omega_I\hat{\pi}_t^I + \omega_N\hat{\pi}_t^{N,G_N}\right)' dS_t \text{ is a } (\check{\mathbb{Q}}^I, \mathbb{F}^I) \text{ martingale.}$$

Recalling the definition of γ in {(2.2)} the above simplifies to

$$\begin{aligned}\overline{\mathbf{M}}_T = \frac{1}{\gamma}&\left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log(\check{Z}_T)\middle|\mathcal{F}^m_0\right] - \log(\check{Z}_T)\right)\\ &- \alpha_U\left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log\left(\ell(T,X_T,H)\right)\middle|\mathcal{F}^m_0\right] - \log\left(\ell(T,X_T,H)\right)\right)\\ &- \alpha_I\left(\left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log(\hat{p}_{C_I}(g-X_T))\middle|\mathcal{F}^m_0\right]\right)_{g=G_I} - \log\left(\hat{p}_{C_I}(G_I - X_T)\right)\right)\\ &- \alpha_N\left(\left(\widetilde{\mathbb{E}}\left[\check{Z}_T\log(\hat{p}_{C_I}(g-X_T))\middle|\mathcal{F}^m_0\right]\right)_{g=G_N} - \log\left(\hat{p}_{C_I}(G_N - X_T)\right)\right).\end{aligned}$$

Now, assume a PCE exists. Then $\omega_U\hat{\pi}^U_t + \omega_I\hat{\pi}^I_t + \omega_N\hat{\pi}^{N,G_N}_t = \Pi$ and {(7.7)} follows. Next, assume {(7.7)}. This gives $\overline{\mathbf{M}}_T = \Pi'(\Psi(X_T) - S_0) = \int_0^T \Pi' dS_t$. By construction in {(7.6)} we know S is a $(\check{\mathbb{Q}}, \mathbb{F}^m)$ martingale, hence $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$ martingale. Thus, we see for all $t \leq T$ that $0 = \check{M}_t := \int_0^t \left(\omega_U\hat{\pi}^U_u + \omega_I\hat{\pi}^I_u + \omega_N\hat{\pi}^{N,G_N}_u - \Pi\right)' dS_u$. Thus, $\check{M}$ is a continuous martingale with quadratic variation

$$0 = \langle\check{M}\rangle_t = \int_0^t \left|\sigma'_u\left(\omega_U\hat{\pi}^U_u + \omega_I\hat{\pi}^I_u + \omega_N\hat{\pi}^{N,G_N}_u - \Pi\right)\right|^2 du.$$

As σ' is non-degenerate $\mathsf{Leb}_{[0,T]}\times\mathbb{P}$ almost surely we have $\omega_U\hat{\pi}^U_t + \omega_I\hat{\pi}^I_t + \omega_N\hat{\pi}^{N,G_N}_t = \Pi$, and hence a PCE exists, finishing the proof. □

Lemma 3.4. *The statements in* (3.24) *hold.*

Proof of Lemma 3.4. Recall (3.15), which states

$$\mathbb{E}^{\mathbb{P}^g}\left[\mathring{Z}^{\check{\mathbb{Q}},g}_T \log\left(\mathring{Z}^{\check{\mathbb{Q}},g}_T\right)\right] = \mathbb{E}^{\check{\mathbb{Q}}}\left[\log\left(\frac{\check{Z}_T}{p^H_T\mathring{p}^{H,g}_T}\right)\right] < \infty.$$

The inequality $x|\log(x)| \leq x\log(x) + 2/e$ implies $\mathbb{E}^{\check{\mathbb{Q}}}\left[|\log(\check{Z}_T/(p^H_T\mathring{p}^{H,g}_T))|\right] < \infty$, which in turn identifies the processes $\theta^{N,g}$ and $\hat{\pi}^{N,g}$ from (3.17). In light of Lemma 3.3 and the above integrability condition, we may apply Proposition 4.6 (with $\mathbb{F} = \mathbb{F}^m$, $\mathbb{P} = \check{\mathbb{Q}}$ and $B = B^{\check{\mathbb{Q}}}$ therein). Part (1) implies θ^{N,G_N}, $\hat{\pi}^{N,G_N}$ are $\mathcal{P}(\mathbb{F}^N)$ measurable, and hence $\mathcal{P}(\mathbb{F}^I)$ measurable, as Lemma 3.3 shows $\mathbb{F}^I = \mathbb{F}^N$. To ease notation, set $\overline{\mathbb{F}}$ as the common filtration. Next, define the measure $\check{\mathbb{Q}}^N$ by

$$\frac{d\check{\mathbb{Q}}^N}{d\mathbb{P}} = \frac{\check{Z}_T}{p^H_T p^{H,G_N}_{N,T}}. \tag{3.25}$$

$\check{\mathbb{Q}}^N$ is the ($\mathbb{F}^m$ to $\overline{\mathbb{F}}$) Martingale preserving measure for $\check{\mathbb{Q}}$ and G_N. As such, $B^{\check{\mathbb{Q}}}$ is a $(\check{\mathbb{Q}}^N, \overline{\mathbb{F}})$ Brownian motion, and from parts (3) and (4) of Proposition 4.6 we know that $\mathbb{Q}^N$ almost surely

$$\int_0^T |\theta_t^{N,G_N}|^2 dt < \infty; \quad \sup_{t\leq T}\left| \int_0^t (\theta_u^{N,G_N})' dB_u^{\check{\mathbb{Q}}} - \left(\int_0^t (\theta_u^{N,g})' dB_u^{\check{\mathbb{Q}}}\right)\Big|_{g=G_N}\right| = 0.$$

As $\mathbb{Q}^I$ is equivalent to $\mathbb{Q}^N$ on $\overline{\mathcal{F}}_T$ it follows that $\hat{\pi}^{N,G_N}$ is S-integrable under $\mathbb{Q}^I$, and from the right equality above that $\mathcal{W}^{\hat{\pi}^{N,G_N}} = \left(\mathcal{W}^{\hat{\pi}^{N,g}}\right)|_{g=G_N}$ under $\mathbb{Q}^I$.

The last thing to show is $\mathcal{W}^{\hat{\pi}^{N,G_N}}$ is a $(\check{\mathbb{Q}}^I, \overline{\mathbb{F}})$-martingale. To this end, we first show that it is a $(\check{\mathbb{Q}}^N, \overline{\mathbb{F}})$ martingale. Indeed,

$$\mathbb{E}^{\check{\mathbb{Q}}^N}\left[\left|\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_N}}\right)\right|\right] = \int_{\mathbb{R}^d} \mathbb{E}^{\check{\mathbb{Q}}}\left[\left|\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,g}}\right)\right|\right] u_N(g)dg, \tag{3.26}$$

where $u_N(g)$ from {(C.1)} is the pdf of G_N. To evaluate this expression set $\chi(g,H) := \mathbb{E}^{\check{\mathbb{Q}}}\left[|\log(\check{Z}_T/(p_T^H \mathring{p}_T^{H,g}))|\big|\mathcal{F}_0^m\right]$. As $\check{Z}_0/(p_0^H \mathring{p}_0^{H,g}) = 1$, $x|\log(x)| \leq x\log(x) + 2/e$, and $\widetilde{\mathbb{E}}\left[\check{Z}_T\big|\mathcal{F}_0^m\right] = 1$, calculations similar to (3.15) show

(3.27)

$$\begin{aligned}\chi(g,H) &\leq \frac{2}{e} + \widetilde{\mathbb{E}}\left[\check{Z}_T \log\left(\frac{\check{Z}_T u(0,X_0,g)}{\hat{p}_I(g - X_T)}\right)\big|\mathcal{F}_0^m\right]; \\ &\leq \frac{2}{e} + \widetilde{\mathbb{E}}\left[\check{Z}_T\log(\check{Z}_T)\big|\mathcal{F}_0^m\right] + (\log(u(0,X_0,g)))^+ + \widetilde{\mathbb{E}}\left[\check{Z}_T(\log(\hat{p}_I(g-X_T)))^-\big|\mathcal{F}_0^m\right].\end{aligned}$$

Using this in (3.26)

(3.28)

$$\begin{aligned}\mathbb{E}^{\check{\mathbb{Q}}^N}\left[\left|\log\left(\frac{\check{Z}_T}{p_T^H \mathring{p}_T^{H,G_N}}\right)\right|\right] &\leq \frac{2}{e} + \widetilde{\mathbb{E}}\left[\check{Z}_T\log(\check{Z}_T)\right] + \int_{\mathbb{R}^d}(\log(u(0,X_0,g)))^+ u_N(g)dg \\ &\quad + \int_{\mathbb{R}^d}\widetilde{\mathbb{E}}\left[\check{Z}_T(\log(\hat{p}_I(g-X_T)))^-\right]u_N(g)dg \quad < \ \infty,\end{aligned}$$

where the last inequality follows from (3.8), {(C.3)} and {(C.4)}(c).

From (3.28), and Lemma 4.8 we know $\mathcal{W}^{\hat{\pi}^{N,G_N}}$ is a $(\check{\mathbb{Q}}^N, \overline{\mathbb{F}})$ martingale. Let us assume for now that

$$\mathbb{E}^{\check{\mathbb{Q}}^I}\left[\left|\mathcal{W}_t^{\hat{\pi}^{N,G_N}}\right|\right] < \infty; \qquad t \leq T. \tag{3.29}$$

The $(\check{\mathbb{Q}}^I, \overline{\mathbb{F}})$-martingale property follows from that under $(\check{\mathbb{Q}}^N, \overline{\mathbb{F}})$ and Lemma 3.3. Indeed, from part (1) of Lemma 3.3 for $t \leq T$

$$\frac{d\check{\mathbb{Q}}^N}{d\check{\mathbb{Q}}^I}\Big|_{\bar{\mathcal{F}}_t} = \frac{d\check{\mathbb{Q}}^N}{d\mathbb{P}}\Big|_{\bar{\mathcal{F}}_t} \times \frac{d\mathbb{P}}{d\check{\mathbb{Q}}^I}\Big|_{\bar{\mathcal{F}}_t} = \frac{p_t^{H,G_I}}{p_{N,t}^{H,G_N}} = \frac{p_0^{H,G_I}}{p_{N,0}^{H,G_N}}. \tag{3.30}$$

As this does not change with t, the martingale property is clear. The last thing to show is (3.29). Using (3.30) and that $\mathcal{W}^{\hat{\pi}^{N,G_N}}$ is a $(\check{\mathbb{Q}}^N,\bar{\mathbb{F}})$ martingale, we find $\mathbb{E}^{\check{\mathbb{Q}}^I}\left[\left|\mathcal{W}_t^{\hat{\pi}^{N,G_N}}\right|\right] \leq \mathbb{E}^{\check{\mathbb{Q}}^I}\left[\left|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\right|\right]$ so (3.29) will hold for $t\leq T$ provided it holds at T. To show this, we use (3.30), Lemma 4.8 and $G_I = \check{G}(G_N,H)$ to obtain

$$\begin{aligned}\mathbb{E}^{\check{\mathbb{Q}}^I}\left[\left|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\right|\right] &= \mathbb{E}^{\check{\mathbb{Q}}^N}\left[\frac{u(0,X_0,\check{G}(G_N,H))|J^{\check{G}}|(G_N,H)}{u_N(G_N)|J^G|(\check{G}(G_N,H),H)}\left|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\right|\right];\\ &= \mathbb{E}^{\check{\mathbb{Q}}}\left[\int_{\mathbb{R}^d}\frac{u(0,X_0,\check{G}(\tilde{g},H))|J^{\check{G}}|(\tilde{g},H)}{|J^G|(\check{G}(\tilde{g},H),H)}\left|\mathcal{W}_T^{\hat{\pi}^{N,\tilde{g}}}\right|d\tilde{g}\right].\end{aligned}$$

From (3.16) we deduce

$$\left|\mathcal{W}_T^{\hat{\pi}^{N,\tilde{g}}}\right| \leq \frac{1}{\gamma_N}\left(\mathbb{E}^{\check{\mathbb{Q}}}\left[\left|\log\left(\frac{\check{Z}_T}{p_T^H\mathring{p}_T^{\tilde{g},H}}\right)\right|\,\middle|\mathcal{F}_0^m\right] + \left|\log\left(\frac{\check{Z}_T}{p_T^H\mathring{p}_T^{\tilde{g},H}}\right)\right|\right).$$

Thus, by first conditioning upon $\mathcal{F}_0^m$ we obtain, using $\chi(g,H)$ from above (c.f. (3.27))

$$\mathbb{E}^{\check{\mathbb{Q}}^I}\left[\left|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\right|\right] \leq \frac{2}{\gamma_N}\mathbb{E}^{\check{\mathbb{Q}}}\left[\int_{\mathbb{R}^d}\frac{u(0,X_0,\check{G}(\tilde{g},H))|J^{\check{G}}|(\tilde{g},H)}{|J^G|(\check{G}(\tilde{g},H),H)}\chi(\tilde{g},H)d\tilde{g}\right].$$

$\check{\mathbb{Q}} = \mathbb{P}$ on $\sigma(H)$ and $H\sim\ell(0,X_0,\cdot)$ under $\mathbb{P}$. Thus

$$\mathbb{E}^{\check{\mathbb{Q}}^I}\left[\left|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\right|\right] \leq \frac{2}{\gamma_N}\int_{\mathbb{R}^d\times\mathcal{R}_H}\frac{u(0,X_0,\check{G}(\tilde{g},h))|J^{\check{G}}|(\tilde{g},h)}{|J^G|(\check{G}(\tilde{g},h),h)}\chi(\tilde{g},h)\ell(0,X_0,h)d\tilde{g}dh.$$

Now, for any appropriately measurable and integrable function f, for $\tilde{g}$ fixed, the substitution $h = H(g,\tilde{g})$ yields

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(g,\tilde{g},H(g,\tilde{g}))dgd\tilde{g} = \int_{\mathbb{R}^d\times\mathcal{R}_H} f(\check{G}(\tilde{g},h),\tilde{g},h)|J^{\check{G}}|(\tilde{g},h)d\tilde{g}dh.$$

At $f(g,\tilde{g},h) = u(0,X_0,g)\chi(\tilde{g},h)\ell(0,X_0,h)/|J^G|(g,h)$, we deduce

$$\mathbb{E}^{\check{\mathbb{Q}}^I}\left[\left|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\right|\right] \leq \frac{2}{\gamma_N}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{u(0,X_0,g)}{|J^G|(g,H(g,\tilde{g}))}\chi(\tilde{g},H(g,\tilde{g}))\ell(0,X_0,H(g,\tilde{g}))d\tilde{g}dg.$$

Keeping g fixed set $h = H(g,\tilde{g})$ so that $\tilde{g} = G(g,h)$ with $d\tilde{g} = |J^G|(g,h)$. This gives

$$\begin{aligned}\mathbb{E}^{\check{\mathbb{Q}}^I}\left[\left|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\right|\right] &\leq \frac{2}{\gamma_N}\int_{\mathbb{R}^d\times\mathcal{R}_H} u(0,X_0,g)\chi(G(g,h),h)\ell(0,X_0,h)dgdh;\\ &= \frac{2}{\gamma_N}\mathbb{E}^{\check{\mathbb{Q}}}\left[\int_{\mathbb{R}^d}\chi(G(g,H),H)u(0,X_0,g)dg\right].\end{aligned}$$

By first conditioning on $\mathcal{F}_0^m$ and using (3.27), then using $\widetilde{\mathbb{E}}\left[\check{Z}_T \middle| \mathcal{F}_0^m\right] = 1$, $H \overset{\widetilde{\mathbb{P}}}{\sim} \ell(0, X, \cdot)$, and $u(0, X_0, \cdot)$ is a pdf we see

(3.31)
$$
\begin{aligned}
\mathbb{E}^{\check{\mathbb{Q}}^I}\left[\left|\mathcal{W}_T^{\hat{\pi}^{N,G_N}}\right|\right] &\leq \frac{4}{e\gamma_N} + \frac{2}{\gamma_N}\widetilde{\mathbb{E}}\left[\check{Z}_T \log\left(\check{Z}_T\right)\right] \\
&\quad + \frac{2}{\gamma_N}\int_{\mathbb{R}^d \times \mathcal{R}_H} \left(\log\left(u(0, X_0, G(g,h))\right)\right)^+ u(0, X_0, g)\ell(0, X_0, h) dg dh \\
&\quad + \frac{2}{\gamma_N}\int_{\mathbb{R}^d} \widetilde{\mathbb{E}}\left[\left(\widetilde{\mathbb{E}}\left[\check{Z}_T \left(\log\left(\hat{p}_I(\tilde{g} - X_T)\right)\right)^- \middle| \mathcal{F}_0^m\right]\right)\Big|_{\tilde{g}=G(g,H)}\right] u(0, X_0, g) dg; \\
&< \infty.
\end{aligned}
$$

Above, the second inequality follows from {(C.3)}, (3.8) and {(C.4)}(d). This verifies (3.29) and hence $\mathcal{W}^{\hat{\pi}^{N,G_N}}$ is a $(\check{\mathbb{Q}}^I, \mathbb{F}^I)$ martingale. Thus, all statements in (3.24) hold. □

4. On initial enlargements

In this section, we collect a number of results for parameter dependent Brownian stochastic integrals in initially enlarged filtrations. Many of these results may be either found in, or deduced from, Stricker and Yor (1978); Amendinger (2000); Gasbarra, Valkeila, and Vostrikova (2006); Esmaeeli and Imkeller (2018), and especially Fontana (2018). We present them for ease of reference.

We take a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ satisfies the usual conditions. There is a d-dimensional $(\mathbb{P}, \mathbb{F})$ Brownian motion B, which has the predictable representation property, but we do not mandate $\mathbb{F} = \mathbb{F}^B$. Next, let $\mathcal{Y} \subseteq \mathbb{R}^m$ be an open set, with Borel sigma-algebra $\mathcal{B}(\mathcal{Y})$. Write $\mathcal{P}(\mathbb{F})$ and $\mathcal{O}(\mathbb{F})$ for the $\mathbb{F}$ predictable and optional sigma algebras. Lastly, for ease of terminology, define

Definition 4.1. $\theta : [0, T] \times \Omega \times \mathcal{Y} \to \mathbb{R}^d$ is *$\mathcal{Y}$-predictable* (respectively *$\mathcal{Y}$-optional*) if θ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{Y})$ (resp. $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{Y})$) measurable.

Let Y be a random variable taking values in $\mathcal{Y}$, and assume

Assumption 4.2. For $t \leq T$, $\mathbb{P}\left[Y \in \cdot \middle| \mathcal{F}_t\right] \sim \mathbb{P}\left[Y \in \cdot\right]$ almost surely. Denote by $p_t^y = p(t, \cdot, y)$ the resultant density and by λ the unconditional law of Y.

Define $\mathbb{G} := \mathbb{F} \vee \mathfrak{s}(Y)$. The first lemma contains three results from Fontana (2018).

Lemma 4.3. *Let Assumption 4.2. Then (1) p is $\mathcal{Y}$-optional; (2) $\mathbb{G}$ satisfies the usual conditions of right-continuity and saturation of $\mathbb{P}$-null sets in $\mathcal{G}_0$; and (3) both p_0^Y/p^Y and $1/p^Y$ are strictly positive $(\mathbb{P}, \mathbb{G})$ martingales with constant expectation 1.*

Proof of Lemma 4.3. Parts (1), (2) and (3) for $1/p^Y$ follow directly from (Fontana, 2018, Lemma 2.3, Lemma 4.2, Proposition 4.4) respectively. As for p_0^Y/p^Y let $0 \leq s < t \leq T$, $A \in \mathcal{F}_s$ and $H \in \mathcal{B}(\mathcal{Y})$. Recalling that λ is the law of Y.

$$\mathbb{E}\left[1_{A_s} 1_{Y \in H} \frac{p_0^Y}{p_t^Y}\right] = \mathbb{E}\left[1_{A_s} \int_H p_0^y \lambda(dy)\right] = \mathbb{E}\left[1_{A_s} 1_{Y \in H} \frac{p_0^Y}{p_s^Y}\right].$$

The first equality follows by conditioning on t and the (reverse) second by conditioning upon s. Taking $A_s = \Omega$ and $H = \mathcal{Y}$ and using (Fontana, 2018, Equation (4.1)) at $f \equiv 1$ shows that $\mathbb{E}\left[\int_{\mathcal{Y}} p_0^y \lambda(dy)\right] = 1$ which finishes the result. □

Remark 4.4. Given Lemma 4.3, it follows from (Jacod, 1985, Section 1) and (Gasbarra, Valkeila, and Vostrikova, 2006, Lemma 4.2) that $\theta = \theta^y$ is $\mathcal{Y}$-predictable if and only if $\theta^Y \in \mathcal{P}(\mathbb{G})$. Additionally, by part (3) above, we may define the martingale preserving measure $\widetilde{\mathbb{P}}^Y$ by either $d\widetilde{\mathbb{P}}_0^Y/d\mathbb{P} = 1/p_T^Y$ or $d\widetilde{\mathbb{P}}^Y/d\mathbb{P} = p_0^Y/p_T^Y$. Note that if $\mathcal{F}_0$ is $\mathbb{P}$-trivial then $\widetilde{\mathbb{P}}_0^Y = \widetilde{\mathbb{P}}^Y$. Next, (Fontana, 2018, Proposition 2.9) proves that B is a $(\widetilde{\mathbb{P}}_0^Y, \mathbb{G})$ Brownian motion with the predictable representation property. Similarly, (Fontana, 2018, Proposition 4.4) implies B is a $(\widetilde{\mathbb{P}}, \mathbb{G})$ Brownian motion. As $d\widetilde{\mathbb{P}}_0^Y/d\widetilde{\mathbb{P}}^Y|_{\mathcal{G}_T} = p_0^Y$ which is $\mathcal{G}_0$ measurable, it follows that B has the predictable representation property under $(\widetilde{\mathbb{P}}, \mathbb{G})$ as well. For technical integrability reasons, it is more convenient for us to work with $\widetilde{\mathbb{P}}^Y$ rather than $\widetilde{\mathbb{P}}_0^Y$.

The first main result concerns martingale representation. To state it, assume

Assumption 4.5. $\phi = \phi(\omega, y)$ is a $\mathcal{F}_T \otimes \mathcal{B}(\mathcal{Y})$ measurable function such that $\mathbb{E}\left[|\phi(\cdot, y)|\right] < \infty$ for each $y \in \mathcal{Y}$.

Next, denote by $\theta = \theta^y$ the process $\theta^y \in \mathcal{P}(\mathbb{F})$ for each $y \in \mathcal{Y}$ and such that

$$M_\cdot^y := \mathbb{E}\left[\phi(\cdot, y) \middle| \mathcal{F}_t\right] = M_0^y + \int_0^\cdot (\theta_u^y)' dB_u. \tag{4.1}$$

We then have the following intuitive result and corollary.

Proposition 4.6. *Let Assumptions 4.2 and 4.5 hold, and let θ be from (4.1). Then (1) θ is $\mathcal{Y}$-predictable, hence $\mathcal{P}(\mathbb{G})$ measurable; (2) The stochastic integral $\int_0^\cdot (\theta_u^y)' dB_u$ is $\mathcal{Y}$-predictable; (3) The stochastic integral $\int_0^\cdot (\theta_u^Y)' dB_u$ is well defined; and (4) $\int_0^\cdot (\theta_u^Y)' dB_u$ and $\left(\int_0^\cdot (\theta_u^y)' dB_u\right)\Big|_{y=Y}$ are indistinguishable.*

Corollary 4.7. *If additionally θ is strictly positive almost surely then the same conclusions hold for $\nu = \nu^y$ defined by*

$$\frac{\phi(\cdot, y)}{\mathbb{E}\left[\phi(\cdot, y)\right]} = \mathcal{E}\left(\int_0^{\cdot} (\nu_u^y)' dB_u\right)_T.$$

Proof of Proposition 4.6. For (1), it follows from (Stricker and Yor, 1978, Proposition 3) that we can take $M = M^y$ in (4.1) to be a cadlag and $\mathcal{B}(\mathcal{Y})$ measurable version of the $\mathbb{F}$-optional projection of $\phi(\cdot, y)$ (see also the proof of (Fontana, 2018, Lemma 4.2)). The result then follows from (Fontana, 2018, Proposition A.1).

(2) is proved in (Stricker and Yor, 1978, Proposition 5) when $\mathbb{E}\left[(\int_0^T |\theta(t, \cdot, y)|^2 dt)^{1/2}\right] < \infty$ (and noting the integral sample paths are continuous). For the general case, set $\theta_n = \theta 1_{|\theta| \leq n}$ and write M^n as the resultant $\mathcal{Y}$-predictable map. Clearly, we have $\mathbb{P}\text{-}\lim_{n,m\to\infty} \int_0^T |\theta_n(t, \cdot, y) - \theta_m(t, \cdot, y)|^2 dt = 0$, and hence by (Karatzas and Shreve, 1991, Prop. 3.2.26) we know $\mathbb{P}\text{-}\lim_{n,m\to\infty} \sup_{t\leq T} |M_n(t, \cdot, y) - M_m(t, \cdot, y)| = 0$. The result follows using (Stricker and Yor, 1978, Proposition 1) with $\mathcal{F}, \mathbb{P}$ there-in being $\mathcal{P}(\mathbb{F})$ and $\mathbb{P} \times \text{Leb}_{[0,T]}$ respectively. For part (3), we first note that by part (1), $\theta^Y \in \mathcal{P}(\mathbb{G})$. Next, as B is a $(\widetilde{\mathbb{P}}^Y, \mathbb{G})$ Brownian motion, hence $(\mathbb{P}, \mathbb{G})$ semi-martingale. Thus, the result will follow if $\mathbb{P}\left[\int_0^T |\theta_u^Y|^2 du < \infty\right] = 1$. By Fubini we know that $1_{\int_0^T |\theta_u^Y|^2 du < \infty} = (1_{\int_0^T |\theta_u^y|^2 du < \infty})|_{y=Y}$, and that $(\omega, g) \to 1_{(\int_0^T |\theta_u^y|^2 du)(\omega) < \infty}$ is $\mathcal{F}_T \otimes \mathcal{B}(\mathcal{Y})$ measurable. Thus, from (Fontana, 2018, Equation (4.1)) we conclude

$$\mathbb{P}\left[\int_0^T |\theta_u^Y|^2 du < \infty\right] = \int_{\mathcal{Y}} \mathbb{E}\left[p_T^y 1_{\int_0^T |\theta_u^y|^2 du < \infty}\right] \lambda(dy) = \int_{\mathcal{Y}} \mathbb{E}\left[p_T^y\right] \lambda(dy) = 1, \tag{4.2}$$

where the last equality follows from (Fontana, 2018, Equation (4.1)) applied to $f \equiv 1$.

That part (4) holds is stated in the proof of (Fontana, 2018, Proposition 4.10) as following from a) an application of the monotone convergence theorem and b) (Stricker and Yor, 1978, Proposition 5) combined with the dominated convergence theorem for stochastic integrals (c.f. (Protter, 2005, IV.Theorem 32)). Part (4) is also implicitly used in the proof of (Amendinger, Imkeller, and Schweizer, 1998, Corollary 2.10). However, for the sake of clarity, we will offer a detailed sketch.

First, assume $\theta(t, \omega, y) = \psi(t, \omega) h(y)$ where $\psi \in \mathcal{P}(\mathbb{F})$, $h \in \mathcal{B}(\mathcal{Y})$ are bounded. Considering integration with respect to the $(\mathbb{P}, \mathbb{F})$-Brownian motion B, it follows that $(\int_0^{\cdot} (\theta_u^y)' dB_u)|_{y=Y} = h(Y) \int_0^{\cdot} \psi(u, \cdot)' dB_u$. Next, considering integration with respect to the $(\widetilde{\mathbb{P}}^Y, \mathbb{G})$-Brownian motion B we have $\int_0^t (\theta_u^Y)' dB_u = h(Y)) \int_0^{\cdot} \psi(u, \cdot)' dB_u$. The result follows by path-continuity. Next, let bounded $\{\theta_n\}$ converge (bounded-ly) to a bounded θ. Write the associated integrals as M_n, M. For each $t \le T$

$$\mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[\left(M(t,\cdot,Y) - \int_0^t \theta(u,\cdot,Y)' dB_u\right)^2\right]$$
$$\le 2\mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[(M(t,\cdot,Y) - M_n(t,\cdot,Y))^2\right] + 2\mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[\left(\int_0^t (\theta_n(u,\cdot,Y) - \theta(u,\cdot,Y))' dB_u\right)^2\right].$$

First,

$$\begin{aligned}
\mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[(M(t,\cdot,Y) - M_n(t,\cdot,Y))^2\right] &= \mathbb{E}\left[\frac{1}{p_t^Y}(M(t,\cdot,Y) - M_n(t,\cdot,Y))^2\right];\\
&= \mathbb{E}\left[\int_{\mathcal{Y}} (M(t,\cdot,y) - M_n(t,\cdot,y))^2 \lambda(dy)\right];\\
&= \mathbb{E}\left[\int_{\mathcal{Y}} \left(\int_0^t |\theta(u,\cdot,y) - \theta_n(u,\cdot,y)|^2 du\right)\lambda(dy)\right];\\
&= \mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[\int_0^t |\theta(u,\cdot,Y) - \theta_n(u,\cdot,Y)|^2 du\right].
\end{aligned}$$

Above we have used the definition of p^Y and the Itô isometry. Similarly

$$\mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[\left(\int_0^t (\theta_n(u,\cdot,Y) - \theta(u,\cdot,Y))' dB_u\right)^2\right] = \mathbb{E}^{\widetilde{\mathbb{P}}^Y}\left[\int_0^t |\theta(u,\cdot,Y) - \theta_n(u,\cdot,Y)|^2 du\right].$$

The bounded convergence theorem implies almost surely for $t \le T$ that $M(t,\cdot,Y) - \int_0^t \theta(u,\cdot,Y)' dB_u = 0$. As θ is bounded, $\int_0^\cdot \theta(u,\cdot,Y)' dB_u$ is a $(\widetilde{\mathbb{P}}^Y, \mathbb{G})$ martingale. But, this implies $M(t,\cdot,Y)$ is also a $(\widetilde{\mathbb{P}}^Y, \mathbb{G})$ martingale. As martingale representation holds with respect to B, we deduce $M(\cdot,\cdot,Y)$ has continuous paths and hence $M(\cdot,\cdot,Y)$ and $\int_0^\cdot \theta(u,\cdot,Y)' dB_u$ are indistinguishable. The monotone class theorem gives the result for bounded θ. We now extend to θ such that $\int_0^T |\theta(u,\cdot,Y)|^2 du < \infty$. For each t, n

$$\begin{aligned}
M(t,\cdot,Y) - \int_0^t \theta(u,\cdot,Y)' dB_u &= \left(\int_0^t (\theta(u,\cdot,y) 1_{|\theta(u,\cdot,y)| \ge n})' dB_u\right)\Big|_{y=Y}\\
&\quad - \int_0^t (\theta(u,\cdot,Y) 1_{|\theta(u,\cdot,Y)| \ge n})' dB_u.
\end{aligned} \tag{4.3}$$

We first handle the right-most term above. By construction of p^Y, for each $\varepsilon > 0$

$$\widetilde{\mathbb{P}}^Y\left[\int_0^T |\theta(u,\cdot,Y)|^2 1_{|\theta(u,\cdot,Y)| \ge n} du \ge \varepsilon\right] = \int_{\mathcal{Y}} \mathbb{E}\left[1_{\int_0^T |\theta(u,\cdot,y)|^2 1_{|\theta(u,\cdot,y)| \ge n} du \ge \varepsilon}\right]\lambda(dy).$$

Since for each $y \in \mathcal{Y}$, $\int_0^T |\theta(u,\cdot,y)|^2 1_{|\theta(u,\cdot,y)| \ge n} du \to 0$ almost surely as $n \uparrow \infty$, two applications of the dominated convergence theorem allow us to conclude that

$\lim_{n\uparrow\infty}\int_0^T |\theta(u,\cdot,Y)|^2 1_{|\theta(u,\cdot,Y)|\geq n} du = 0$ in $\widetilde{\mathbb{P}}^Y$ probability. Therefore, by (Karatzas and Shreve, 1991, Prop. 3.2.26) we know that in $\widetilde{\mathbb{P}}^Y$ probability

$$\lim_{n\uparrow\infty}\sup_{t\leq T}\left|\int_0^t (\theta(u,\cdot,Y)1_{|\theta(u,\cdot,Y)|\geq n})' dB_u\right| = 0.$$

As for the first term on the right side of (4.3) set $M_n(t,\cdot,y) := \int_0^t (\theta(u,\cdot,y)1_{|\theta(u,\cdot,y)|\geq n})' dB_u$. Since for each $y\in\mathcal{Y}$, $\int_0^T |\theta(u,\cdot,y)|^2 1_{|\theta(u,\cdot,y)|\geq n} du$ converges to 0 almost surely, we again deduce from (Karatzas and Shreve, 1991, Prop 3.2.26) that in $\mathbb{P}$ probability $\sup_{t\leq T}|M_n(t,\cdot,y)|\to 0$. As M_n is $\mathcal{Y}$-optional,

$$\widetilde{\mathbb{P}}^Y\left[\sup_{t\leq T}|M_n(t,\cdot,Y)|\geq\varepsilon\right] = \int_Y \mathbb{E}\left[1_{\sup_{t\leq T}|M_n(t,\cdot,y)|\geq\varepsilon}\right]\lambda(dy),$$

so that $\sup_{t\leq T}|M_n(t,\cdot,Y)|\to 0$ in $\widetilde{\mathbb{P}}^Y$ probability. Thus, by taking subsequences where the convergence takes place almost surely $\mathbb{Q}^G$ and hence $\mathbb{P}$, we deduce from (4.3) that $\sup_{t\leq T}|M(t,\cdot,Y) - \int_0^t \theta(u,\cdot,Y)' dB_u| = 0$ almost surely, finishing the result. □

Proof of Corollary 4.7. It is clear that $\nu = \theta/M$. Thus, by the results on θ above (in particular the connection the proof of (4) which proved indistinguishability for general θ), it suffices to prove that $\int_0^T |\nu_u^Y|^2 du < \infty$ almost surely. But, this will follow provided $\inf_{t\leq T} M_t^Y > 0$ almost surely. But, this latter fact follows using the same calculations as in (4.2), but now for the random variable $1_{\inf_{t\leq T} M_t^y > 0}$, which is almost surely 1 for all y since M^y has continuous paths. □

Lastly, we relate $(\mathbb{Q}^{\mathbb{G}},\mathbb{G})$ and $(\mathbb{Q},\mathbb{F})$ conditional expectations, where $\mathbb{Q}$ is a measure on $\mathcal{F}_T$, and $\mathbb{Q}^{\mathbb{G}}$ is built from $\mathbb{Q}$ in a similar manner to how $\widetilde{\mathbb{P}}^Y$ was built from $\mathbb{P}$.

Lemma 4.8. *Let Z be a strictly positive $(\mathbb{P},\mathbb{F})$ martingale with $\mathbb{E}[Z_0] = 1$. Define $\mathbb{Q}$ via $d\mathbb{Q}/d\mathbb{P} := Z_T$ and $Z^{\mathbb{G}}_\cdot := Z_\cdot/p^Y_\cdot$. Then $Z^{\mathbb{G}}$ is a $(\mathbb{P},\mathbb{G})$ martingale. Next, define $\mathbb{Q}^{\mathbb{G}}$ by $d\mathbb{Q}^{\mathbb{G}}/d\mathbb{P} = Z^{\mathbb{G}}_T$. Let $0\leq s<t\leq T$, and let ϕ be $\mathcal{F}_t$ measurable, taking values in $G\subseteq\mathbb{R}^n$; and $f: G\times\mathcal{Y}\to\mathbb{R}$ be a measurable function such that either a) f is non-negative; or b) that $\mathbb{E}^{\mathbb{Q}}[|f(\phi,y)|] < \infty$ for each $y\in\mathcal{Y}$ as well as $\int_{\mathcal{Y}}\mathbb{E}^{\mathbb{Q}}[|f(\phi,y)|]\lambda(dy) < \infty$. Then*

$$\mathbb{E}^{\mathbb{Q}^{\mathbb{G}}}\left[f(\phi,Y)\middle|\mathcal{G}_s\right] = \left(\mathbb{E}^{\mathbb{Q}}\left[f(\phi,y)\middle|\mathcal{F}_s\right]\right)\Big|_{y=\mathcal{Y}}.$$

Proof of Lemma 4.8. Let $0 \leq s < t \leq T$, $A_s \in \mathcal{F}_s$, $H \in \mathcal{B}(\mathcal{Y})$ and denote by λ the law of Y. We have

$$\begin{aligned}\mathbb{E}\left[1_{A_s}1_{Y\in H}Z_t^{\mathbb{G}}\right] &= \mathbb{E}\left[1_{A_s}1_{Y\in H}Z_t\frac{1}{p_t^Y}\right] = \int_H \mathbb{E}\left[1_{A_s}Z_t 1\right]\lambda(dy);\\ &= \int_H \mathbb{E}\left[1_{A_s}Z_s\right]\lambda(dy) = \mathbb{E}\left[1_{A_s}1_{Y\in H}Z_s^{\mathbb{G}}\right].\end{aligned}$$

Taking the above for $A_s = \Omega, H = \mathcal{Y}$ we see $\mathbb{E}\left[Z_t^{\mathbb{G}}\right] = \int_{\mathcal{Y}}\mathbb{E}\left[Z_t\right]\lambda(dy) = \int_{\mathcal{Y}}\mathbb{E}\left[Z_0\right]\lambda(dy) = 1$. Here, we have used the martingale property for Z and $\mathbb{E}\left[Z_0\right] = 1$. The martingale property readily follows. As for the conditional expectation equality, if f is not non-negative, the condition $\int_{\mathcal{Y}}\mathbb{E}^{\mathbb{Q}}\left[|f(\phi,y)|\right]\lambda(dy) < \infty$ implies $\mathbb{E}^{\mathbb{Q}^{\mathbb{G}}}\left[|f(\phi,Y)|\right] < \infty$ so the conditional expectation is well defined. Next, let $A_s \in \mathcal{F}_s$ and $H \in \mathcal{B}(\mathcal{Y})$. Set $\chi_s^t(y) := \mathbb{E}^{\mathbb{Q}}\left[f(\phi,y)\middle|\mathcal{F}_s^B\right]$. Note that $(\omega,y)\to\chi_s^t(y)$ is $\mathcal{F}_s\times\mathcal{B}(\mathcal{Y})$ measurable, and hence $\chi_s^t(Y)$ is $\mathcal{G}_s$ measurable. As $Z^{\mathbb{G}}$ is a $(\mathbb{P},\mathbb{G})$ martingale,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{\mathbb{G}}}\left[1_{A_s}1_{Y\in H}\chi_s^t(Y)\right] &= \mathbb{E}\left[1_{A_s}1_{Y\in H}\chi_s^t(Y)\frac{Z_s}{p_s^Y}\right] = \mathbb{E}\left[1_{A_s}Z_s\mathbb{E}\left[1_{Y\in H}\chi_s^t(Y)\frac{1}{p_s^Y}\middle|\mathcal{F}_s^B\right]\right];\\ &= \mathbb{E}\left[1_{A_s}Z_s\int_H\chi_s^t(y)1\lambda(dy)\right] = \int_H\mathbb{E}\left[1_{A_s}Z_s\chi_s^t(y)\right]\lambda(dy);\\ &= \int_H\mathbb{E}\left[1_{A_s}Z_t f(\phi,y)\right]\lambda(dy) = \mathbb{E}\left[1_{A_s}Z_t\int_H\frac{1}{p_t^y}p_t^y f(\phi,y)\lambda(dy)\right];\\ &= \mathbb{E}\left[1_{A_s}1_{Y\in H}\frac{Z_t}{p_t^Y}f(\phi,Y)\right] = \mathbb{E}^{\mathbb{Q}^{\mathbb{G}}}\left[1_{A_s}1_{Y\in H}f(\phi,Y)\right].\end{aligned}$$

□

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