A Additional Discussions and Results

A.1 Equivalence between the two-dimensional value model and the Pareto weight model

In this appendix, we establish an equivalence between *(i)* our "two-dimensional" model, in which the designer maximizes total value (VAL) over feasible mechanisms according to Definition 3 and *(ii)* a "one-dimensional" model in which agents only report their rate of substitution r and the designer maximizes weighted surplus (with Pareto weights λ_j) according to (VAL'). While we only need one direction of the equivalence to justify the derivation of optimal mechanisms in Section 4, we demonstrate the full equivalence to show that we could just as well start our analysis with the one-dimensional model (with Pareto weights given as a primitive of the model) and our conclusions would remain identical.

To simplify notation, we use $(\bar{X}_B,\, \bar{X}_S,\, \bar{T}_B,\, \bar{T}_S)$ to denote a generic (direct) mechanism eliciting (v^K, v^M) and $(X_B,\, X_S,\, T_B,\, T_S)$ to denote a generic (direct) mechanism eliciting r. Formally, a mechanism $(X_B,\, X_S,\, T_B,\, T_S)$ is feasible in the one-dimensional model if for all r, $\hat{r}$:

$$\begin{aligned}
X_B(r)r - T_B(r) &\geq X_B(\hat{r})r - T_B(\hat{r}),\\
-X_S(r)r + T_S(r)r &\geq -X_S(\hat{r})r + T_S(\hat{r}),\\
X_B(r)r - T_B(r) &\geq 0,\\
-X_S(r)r + T_S(r) &\geq 0,\\
\int_{\underline{r}_B}^{\bar{r}_B} X_B(r)\mu dG_B(r) &= \int_{\underline{r}_S}^{\bar{r}_S} X_S(r)dG_S(r),\\
\int_{\underline{r}_B}^{\bar{r}_B} T_B(r)\mu dG_B(r) &\geq \int_{\underline{r}_S}^{\bar{r}_S} T_S(r)dG_S(r).
\end{aligned}$$

A feasible mechanism $(X_B,\, X_S,\, T_B,\, T_S)$ is optimal in the one-dimensional model if it maximizes (VAL') among all feasible mechanisms.

Theorem 8. *If a mechanism $(\bar{X}_B,\, \bar{X}_S,\, \bar{T}_B,\, \bar{T}_S)$ is feasible (resp. optimal) in the two-dimensional model, then there exists a payoff-equivalent mechanism $(X_B,\, X_S,\, T_B,\, T_S)$ eliciting one-dimensional reports that is feasible (resp. optimal) in the one dimensional* *model with G_j equal to the distribution of v^K/v^M under F_j and λ_j given by*

$$\lambda_j(r) = \mathbb{E}^j\left[v^M \mid \frac{v^K}{v^M} = r\right]. \tag{A.1}$$

Conversely, if a mechanism (X_B, X_S, T_B, T_S) is feasible (resp. optimal) in the one-dimensional model, then there exists a joint distribution F_j of (v^K, v^M) such that this mechanism (with agents reporting v^K/v^M) is feasible (resp. optimal) in the two-dimensional model, v^K/v^M has distribution G_j, and (A.1) holds.

Proof. We establish Theorem 8 in three steps:

1. We show that, without loss of generality, an incentive-compatible mechanism in the two-dimensional model only elicits information about the rate of substitution, v^K/v^M; thus, the space of feasible mechanisms is effectively the same in both settings.

2. We argue that the total value function (VAL) corresponds exactly to the objective function (VAL') with Pareto weights $\lambda_j(r)$ taken to be the expected value of v^M conditional on observing a rate of substitution $r = v^K/v^M$.

3. As a consequence, if G_j is the distribution of v^K/v^M under F_j, and Pareto weights are defined as in Step 2, the same mechanism is optimal in both settings.

Step 1. We first formalize the idea that although agents have two-dimensional types, it is without loss of generality to consider mechanisms that only elicit information about the rate of substitution. While it is clear that the rate of substitution fully describes agents' behavior, it could hypothetically be possible that the designer would elicit more information by offering different combinations of trade probabilities and transfers among which the agent is indifferent; we show, however, that this is only possible for a measure-zero set of agents' types, and thus cannot strictly improve the designer's objective.

Lemma 2. *If $(\bar{X}_B, \bar{X}_S, \bar{T}_B, \bar{T}_S)$ is an incentive-compatible mechanism, then there exists a mechanism (X_B, X_S, T_B, T_S) such that $\bar{X}_j(v^K, v^M) = X_j(v^K/v^M)$ and $\bar{T}_j(v^K, v^M) = T_j(v^K/v^M)$ for almost all (v^K, v^M) and $j \in \{B, S\}$.*

We prove Lemma 2 at the end of Appendix A.1.[33] Thanks to the lemma, and the assumption that there are no mass points in the distribution of values, we can assume (without loss of optimality) that agents report their rates of substitution v^K/v^M in the two-dimensional model. Consequently, by direct inspection of the definition, the space of feasible mechanisms is the same in both models.

Step 2. Suppose that the distribution F_j and the weights $\lambda_j(r)$ are such that: $\Lambda_j = \mathbb{E}^j[v^M]$, for $j \in \{B, S\}$, and $\lambda_j(r)$ is given by (A.1). Moreover, let G_j be the distribution of the random variable v^K/v^M when (v^K, v^M) is distributed according to F_j. Then, using Step 1, the objective functions (VAL) and (VAL') become identical:

$$\mu\mathbb{E}^B\left[X_B\left(\frac{v^K}{v^M}\right)v^K - T_B\left(\frac{v^K}{v^M}\right)v^M\right] + \mathbb{E}^S\left[-X_S\left(\frac{v^K}{v^M}\right)v^K + T_S\left(\frac{v^K}{v^M}\right)v^M\right]$$
$$= \mu\mathbb{E}^B\left[\underbrace{\mathbb{E}^B\left[v^M \mid r\right]}_{\lambda_B(r)}\underbrace{\left[X_B(r)\,r - T_B(r)\right]}_{U_B(r)}\right] + \mathbb{E}^S\left[\underbrace{\mathbb{E}^S\left[v^M \mid r\right]}_{\lambda_S(r)}\underbrace{\left[-X_S(r)\,r + T_S(r)\right]}_{U_S(r)}\right].$$

Step 3. The first part of Theorem 8 follows directly from preceding arguments. To prove the second part, we have to show that for any (fixed) G_j and $\lambda_j(r)$, there exists a distribution F_j of (v^K, v^M) that induces that G_j and $\lambda_j(r)$. The proof is simple: Fixing the random variable r (on some probability space) with distribution $G_j(r)$, define random variables $v^K = r\lambda_j(r)$ and $v^M = \lambda_j(r)$. It is clear that the distribution of v^K/v^M is the same as that of r because these random variables are equal. Moreover, by construction, equation (A.1) must hold. $\square$

Proof of Lemma 2

We start with the following result that provides a key step in the proof.

Lemma 3. *Let $X(\theta_1, \theta_2)$ be a function defined on $[\underline{\theta}_1, \bar{\theta}_1] \times [\underline{\theta}_2, \bar{\theta}_2]$, with $\underline{\theta}_1, \underline{\theta}_2 \geq 0$, and assume that $X(\theta_1, \theta_2)$ is non-decreasing in θ_1/θ_2, that is*

$$\frac{\theta_1}{\theta_2} > \frac{\theta_1'}{\theta_2'} \implies X(\theta_1, \theta_2) \geq X(\theta_1', \theta_2').$$

[33]Lemma 4 of Che et al. (2013)—who study a different economic problem—is mathematically equivalent to Lemma 2; we nevertheless provide a proof for completeness.

Then, there exists a non-decreasing function $x : [\underline{\theta}_1/\bar{\theta}_2,\, \bar{\theta}_1/\underline{\theta}_2] \to \mathbb{R}$ *such that* $X(\theta_1,\, \theta_2) = x(\theta_1/\theta_2)$ *almost everywhere.*

Proof. Consider $Y(r,\, \theta_2) = X(r\theta_2,\, \theta_2)$. For small enough $\epsilon > 0$ and almost all $r \in [\underline{\theta}_1/\bar{\theta}_2,\, \bar{\theta}_1/\underline{\theta}_2]$,

$$Y(r+\epsilon,\, \theta_2) \geq Y(r,\, \theta_2'),\ \forall \theta_2,\, \theta_2',$$

by assumption. Because $Y(r,\, \theta_2)$ is non-decreasing in r for every θ_2, it is continuous in r almost everywhere. Thus, for almost all r we obtain

$$Y(r,\, \theta_2) \geq Y(r,\, \theta_2'),\ \forall \theta_2,\, \theta_2';$$

this, however, means that $Y(r,\, \theta_2) = x(r)$ for almost all r (does not depend on θ_2), for some function x, that is moreover non-decreasing. Thus, $X(r\theta_1,\, \theta_2) = x(r)$ for almost all r. Therefore,

$$X(\theta_1,\, \theta_2) = X\left(\frac{\theta_1}{\theta_2}\,\theta_2,\, \theta_2\right) = x\left(\frac{\theta_1}{\theta_2}\right)$$

almost everywhere, as desired. □

We now show that incentive-compatibility for buyers implies that $\bar{X}_B(v^K,\, v^M) = X_B(v^K/v^M)$ for some non-decreasing X_B. The argument for sellers is identical, and the statement for transfer rules follows immediately from the payoff equivalence theorem.

Incentive-compatibility means that for all $(v^K,\, v^M)$ and $(\hat{v}^K,\, \hat{v}^M)$ in the support of F_B we have

$$\bar{X}_B(v^K,\, v^M)\frac{v^K}{v^M} - \bar{T}_B(v^K,\, v^M) \geq \bar{X}_B(\hat{v}^K,\, \hat{v}^M)\frac{v^K}{v^M} - \bar{T}_B(\hat{v}^K,\, \hat{v}^M), \tag{A.2}$$

as well as

$$\bar{X}_B(\hat{v}^K,\, \hat{v}^M)\frac{\hat{v}^K}{\hat{v}^M} - \bar{T}_B(\hat{v}^K,\, \hat{v}^M) \geq \bar{X}_B(v^K,\, v^M)\frac{\hat{v}^K}{\hat{v}^M} - \bar{T}_B(v^K,\, v^M). \tag{A.3}$$

Putting (A.2) and (A.3) together, we have

$$\left(\bar{X}_B(v^K,\, v^M) - \bar{X}_B(\hat{v}^K,\, \hat{v}^M)\right)\left(\frac{v^K}{v^M} - \frac{\hat{v}^K}{\hat{v}^M}\right) \geq 0.$$

It follows that
$$\frac{v^K}{v^M} > \frac{\hat{v}^K}{\hat{v}^M} \implies \bar{X}_B(v^K,\, v^M) \geq \bar{X}_B(\hat{v}^K,\, \hat{v}^M).$$
By Lemma 3, it follows that there exists a non-decreasing $X_B(\cdot)$ such that

$$\bar{X}_B(v^K,\, v^M) = X_B(v^K/v^M)$$

almost everywhere, which finishes the proof.

A.2 Why a factor of 2 in the definition of inequality?

In this appendix, we offer intuition for why 2 is the threshold separating low and high same-side inequality—that is, why rationing may be part of an optimal mechanism only when the trader with the lowest rate of substitution has a conditional value for money more than *twice* the average value. We focus on the seller side of the market, although an analogous intuition holds for the buyer side, as well.

With high seller-side inequality, Proposition 1 indicates that rationing is optimal at small volumes of trade (if the budget constraint is not too tight). To simplify notation, we assume that $\underline{r}_S = 0$, and consider the welfare associated with posting a small price $p \approx 0$. As p is small, we can treat $\lambda_S(r)$ as being approximately constant—equal to $\lambda_S(0)$—for $r \in [0,\, p]$.

If the budget constraint is not binding, then the opportunity cost of a unit of money spent on purchases of the object is the marginal value of the lump-sum transfer, Λ_S. Thus, the welfare gain from setting price p is

$$G_0 \equiv \int_0^p \lambda_S(r)(p-r)dG_S(r) \approx \lambda_S(0)g_S(0)\int_0^p (p-r)dr,$$

while the (opportunity) cost is

$$C_0 \equiv \Lambda_S \cdot pG_S(p).$$

Now, suppose that the designer considers introducing rationing by raising the price to $p+\epsilon$ but keeping the quantity fixed, for some small ϵ. The gain is now

$$G_1 \equiv \frac{G_S(p)}{G_S(p+\epsilon)}\int_0^{p+\epsilon} \lambda_S(r)[p+\epsilon-r]dG_S(r) \approx \lambda_S(0)g_S(0)\int_0^{p+\epsilon}(p+\epsilon-r)\frac{p}{p+\epsilon}dr,$$

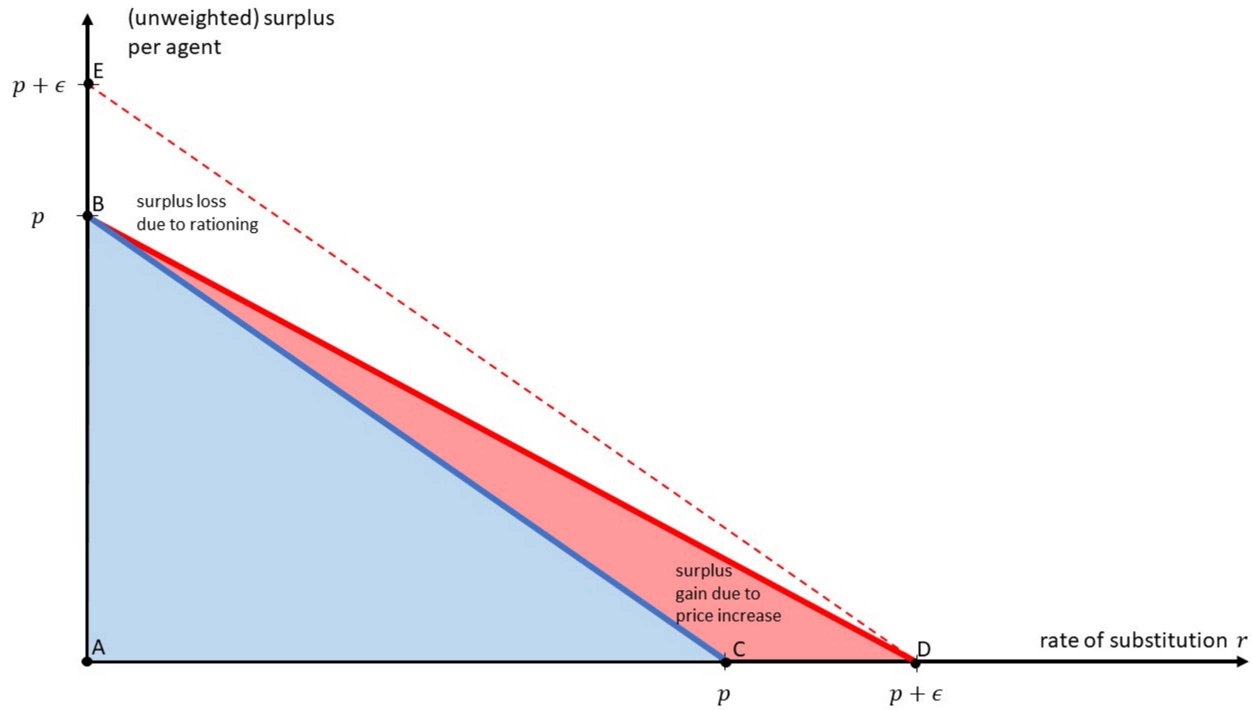

Figure A.1: The surplus (gross of lump-sum transfers) from posting a price p (blue triangle ABC) versus from rationing at a price $p+\epsilon$ (red triangle ABD).

where $\frac{G_S(p)}{G_S(p+\epsilon)}$ is the rationing coefficient, and the new opportunity cost is

$$C_1 \equiv \Lambda_S \cdot (p+\epsilon)G_S(p) = C_0 + \epsilon\Lambda_S g_S(0)p.$$

Rationing is optimal when the change in gains exceeds the change in costs:

$$\Delta G \equiv \underbrace{\lambda_S(0)}_{\text{value for money}} \underbrace{g(0)p}_{\text{mass}} \underbrace{\frac{1}{2}\,\epsilon}_{\text{per agent surplus}} > \Delta C \equiv \underbrace{\Lambda_S}_{\text{value for money}} \underbrace{g(0)p}_{\text{mass}} \underbrace{\epsilon}_{\text{per agent cost}},$$

that is, when $\lambda_S(0) > 2\Lambda_S$. Intuitively, increasing the price received by sellers by ϵ requires raising ϵ in additional revenue. But when the designer increases price by ϵ, half of the resulting surplus is wasted because of inefficient rationing. Thus, for the switch to rationing to be socially optimal, it has to be that the agents who receive the extra ϵ of money value it at least twice as much as do the agents who give it up.

This intuition is illustrated in Figure A.1. The surplus G_0 associated with price p is given by the blue triangle ABC. The dotted red triangle AED illustrates the hypothetical surplus associated with raising the price to $p+\epsilon$ without rationing—which increases surplus by an amount proportional to ϵ (up to terms that are second-order in ϵ). With rationing, the actual surplus is increased by an amount proportional to $\frac{\epsilon}{2}$ and given by the area of the solid red triangle ABD (the seller with rate of substitution 0 is exactly indifferent between receiving a price p for sure and receiving the price $p+\epsilon$ with probability $\frac{p}{p+\epsilon}$ under rationing). The white area between the solid red triangle ABD and the dotted red triangle AED represents the surplus loss due to inefficient rationing. The figure depicts unweighted surplus—the actual contribution of the triangular areas to welfare is given by multiplying the area by the conditional value for money, which is approximately $\lambda_S(0)$ when p is small. Rationing is optimal when

$$\frac{\lambda_S(0)\cdot\epsilon}{2}$$

exceeds the per-agent change in costs associated with the price increase from p to $p+\epsilon$, which is

$$\Lambda_S\cdot\epsilon.$$

The intuition just presented illustrates, in particular, that the threshold of 2 does not depend on our uniform distribution assumption. Indeed, our reasoning only relied on local (first-order) changes, so all the calculations remain approximately valid for any distribution G_S that has a positive continuous density around its lower bound $\underline{r}_S$. For small changes in the price, the region of the surplus change is approximately a triangle, and hence the factor of 2 comes out of the formula for the area of a triangle.

B Proofs Omitted from the Main Text

In this section, we prove the results from Section 3–Section 5. Because the results in Section 3 are mostly corollaries of the general results derived in Section 5, we first prove the results of Section 4, then those of Section 5, and lastly those of Section 3.

B.1 Proof of Lemma 1

Our optimization problem is an infinite-dimensional linear program: To use a Lagrangian approach, we need to check that a relevant qualification constraint is satisfied. Indeed, constraint (4.5) satisfies the generalized Slater condition (see, e.g., Proposition 2.106 and Theorem 3.4 of Bonnans and Shapiro, 2000).[34] Thus, an approach based on putting a Lagrange multiplier $\alpha \geq 0$ on the constraint (4.5) is valid (strong duality holds). Moreover, we can assume without loss of generality that constraint (4.5) binds at the optimal solution (because G_j admits a density, it follows that there exists a positive measure of buyers and sellers with strictly positive value for good M). This means that the problem (4.3)–(4.5) is equivalent to the following statement: There exists $\alpha^\star \geq 0$ such that the solution to the problem

$$\max\left\{\int_0^1 \phi_B^{\alpha^\star}(q)d(\mu H_B(q)) + \int_0^1 \phi_S^{\alpha^\star}(q)dH_S(q)\right\} \tag{B.1}$$

over H_S, $H_B \in \Delta([0,\,1])$, $\underline{U}_B$, $\underline{U}_S \geq 0$, subject to

$$\int_0^1 qd(\mu H_B(q)) = \int_0^1 qdH_S(q) \tag{B.2}$$

satisfies constraint (4.5) with equality.

The value of the problem (B.1)–(B.2) can be computed by parameterizing $Q = \int_0^1 qd(\mu H_B(q))$ and noticing that for a fixed Q, the choice of the optimal H_S is formally equivalent to choosing a distribution of posterior beliefs in a Bayesian persuasion problem with two states, where equation (B.2) is the Bayes plausibility constraint. Hence, by Aumann et al. (1995) or Kamenica and Gentzkow (2011), the optimal distribution $H_S^\star$ yields the value of the concave closure of $\phi_S^{\alpha^\star}$ at Q. Similarly, the optimal distribution $H_B^\star$ yields the value of the concave closure of $\mu\phi_B^{\alpha^\star}$ at Q/μ. Optimizing the value of the unconstrained problem $\mathrm{co}\left(\phi_S^{\alpha^\star}\right)(Q) + \mu\,\mathrm{co}\left(\phi_B^{\alpha^\star}\right)(Q/\mu)$ over Q, $\underline{U}_B$, $\underline{U}_S \geq 0$ yields the optimal solution to the original problem if constraint (4.5) holds with equality at that solution. This gives the first part of the lemma.

Conversely, if $H_B^\star$ and $H_S^\star$ are optimal, then constraints (4.4)–(4.5) must bind. Optimality of $H_j^\star$ implies that the value of ϕ_j^α at the optimum must be equal to its concave closure. We can define $Q = \int_0^1 qdH_S(q)$, and it must be that there exists $\alpha^\star \geq 0$ such that Q maximizes (B.1); this yields the second part of the lemma.

[34] Roughly, this condition requires the feasible set to have an interior point. This can be easily guaranteed for our problem by endowing the space of distributions with, e.g., the weak* topology.

B.2 Completion of the proof of Theorem 1

First, we determine the optimal lump-sum transfers. Lemma 1 requires that the problem

$$\max_{Q\in[0,1],\,\underline{U}_B,\,\underline{U}_S\geq 0}\left\{\mathrm{co}\left(\phi_S^{\alpha^\star}\right)(Q)+\mu\,\mathrm{co}\left(\phi_B^{\alpha^\star}\right)(Q/\mu)\right\}$$

has a solution, and this restricts the Lagrange multiplier to satisfy $\alpha^\star\geq\max\{\Lambda_S,\,\Lambda_B\}$. Indeed, in the opposite case, it would be optimal to set $\underline{U}_j=\infty$ for some j and this would clearly violate constraint (4.5). When $\Lambda_B=\Lambda_S$, it is either optimal to set $\alpha^\star>\Lambda_S=\Lambda_B$ and satisfy (4.5) with equality and $\underline{U}_S=\underline{U}_B=0$ (in which case there is no revenue and no redistribution), or to set $\alpha^\star=\Lambda_S=\Lambda_B$ and $\underline{U}_S=\underline{U}_B\geq 0$ to satisfy (4.5) with equality (in which case the revenue is redistributed to both buyers and sellers as an equal lump-sum payment).[35] When $\Lambda_B>\Lambda_S$, by similar reasoning, $\underline{U}_S$ must be 0, and $\underline{U}_B\geq 0$ is chosen to satisfy (4.5). When $\Lambda_S>\Lambda_B$, it is the seller side that obtains the lump-sum payment that balances the budget (4.5). In short, we can write the conditions for optimality of $\underline{U}_S$ and $\underline{U}_B$ as (ignoring the constraint (4.5) for now)

$$\underline{U}_S\geq 0,\;\underline{U}_S(\alpha^\star-\Lambda_S)=0; \tag{B.3}$$

$$\underline{U}_B\geq 0,\;\underline{U}_B(\alpha^\star-\Lambda_B)=0. \tag{B.4}$$

Next, we consider the optimal lotteries $H_S^\star$ and $H_B^\star$. From Lemma 1, we know that each optimal lottery $H_j^\star$ concavifies a one-dimensional function ϕ_S^α while satisfying a single linear constraint (4.5). Therefore, by Carathéodory's Theorem, we can assume without loss of generality that $H_j^\star$ is supported on at most three points (an analogous mathematical observation in the context of persuasion was first made by Le Treust and Tomala, 2019, and is further generalized and explained in Doval and Skreta, 2018). We argue next that the dimension of the optimal pair of lotteries $(H_B^\star,\,H_S^\star)$ can be further reduced.

We denote $\psi_B(q):=\int_{G_B^{-1}(1-q)}^{\bar{r}_B}J_B(r)g_B(r)dr$ and $\psi_S(q):=\int_{\underline{r}_S}^{G_S^{-1}(q)}J_S(r)g_S(r)dr$. We then let $\mathrm{supp}(H_j^\star)=\{q_j^1,\,q_j^2,\,q_j^3\}$ with $q_j^1\leq q_j^2\leq q_j^3$. Observe that because the distribution $H_j^\star$ concavifies $\phi_j^{\alpha^\star}(q)$, it must be that $\mathrm{co}(\phi_j^{\alpha^\star})(q)$ is affine on the convex hull of the support of $H_j^\star$. Moreover, because the volume of trade $Q\equiv\int_0^1 q\,dH_s^\star(q)$

[35] Of course, in this case, the surplus can also be redistributed only to the sellers, or only to the buyers, as long as condition (4.5) holds.

maximizes the concavified Lagrangian $\text{co}\left(\phi_S^{\alpha^\star}\right)(q) + \mu\,\text{co}\left(\phi_B^{\alpha^\star}\right)(q/\mu)$ over q, it follows that the Lagrangian is constant in q on $\text{supp}(H_B^\star) \cap \text{supp}(H_S^\star)$, that is, on $[\underline{q},\, \bar{q}] \equiv [\max\{q_S^1,\, q_B^1\},\, \min\{q_S^3,\, q_B^3\}]$. Thus, $Q \in [\underline{q},\, \bar{q}]$ and any volume of trade between $\underline{q}$ and $\bar{q}$ is optimal.

The above reasoning, in particular the last observation, implies that the following linear system admits a solution (here, the solution ν_j^i represents the realization probabilities of each q_j^i, while $\underline{U}_S$ and $\underline{U}_B$ satisfy (B.3) and (B.4)):

$$\mu \sum_{i=1}^{3} \nu_B^i q_B^i = \sum_{i=1}^{3} \nu_S^i q_S^i, \tag{B.5}$$

$$\mu \sum_{i=1}^{3} \nu_B^i \psi_B(q_B^i) = \sum_{i=1}^{3} \nu_S^i \psi_S(q_S^i) + \underline{U}_S + \mu \underline{U}_B, \tag{B.6}$$

$$\sum_{i=1}^{3} \nu_B^i = 1,\ \sum_{i=1}^{3} \nu_S^i = 1, \tag{B.7}$$

$$\nu_B^1,\, \nu_B^2,\, \nu_B^3,\, \nu_S^1,\, \nu_S^2,\, \nu_S^3 \geq 0, \tag{B.8}$$

$$\underline{q} \leq \sum_{i=1}^{3} \nu_S^i q_S^i \leq \bar{q}, \tag{B.9}$$

where (B.5) and (B.6) correspond to binding constraints (4.4) and (4.5), (B.7)–(B.8) state that we have well-defined probability measures, and (B.9) makes sure that the Lagrangian is maximized. From Lemma 1, any solution $(\underline{U}_S,\, \underline{U}_B,\, \nu)$ to the linear system (B.3)–(B.9) constitutes a solution to our original problem (by defining $H_j^\star$ as putting a probability weight ν_j^i on q_j^i for all i and j). We can now establish the key claim.

Claim 2. *Either:*

- *there exists a solution* $(\underline{U}_S,\, \underline{U}_B,\, H_B^\star, H_S^\star)$ *to* (B.3)–(B.9) *with* $\underline{U}_S = \underline{U}_B = 0$ *and*

$$|\,\text{supp}(H_B^\star)| + |\,\text{supp}(H_S^\star)| \leq 4;\ \textit{or} \tag{B.10}$$

- *there exists a solution* $(\underline{U}_S,\, \underline{U}_B,\, H_B^\star, H_S^\star)$ *to* (B.3)–(B.9) *with*

$$|\,\text{supp}(H_B^\star)| + |\,\text{supp}(H_S^\star)| \leq 3. \tag{B.11}$$

Proof. We consider two cases. First suppose that $\alpha^\star = \Lambda_j$ for some j. For concreteness and without loss of generality, we let $j = S$ and set $\underline{U}_B = 0$; note that this automatically satisfies (B.4). Then, constraints (B.3) – (B.4) reduce to

$$\underline{U}_S \geq 0. \tag{B.12}$$

The linear system (B.5)–(B.9), (B.12) has four equality constraints and seven free variables (six variable in the vector ν and $\underline{U}_S$), and admits a solution. By the Fundamental Theorem of Linear Algebra, there exists a solution in which seven constraints in the problem (B.5)–(B.9), (B.12) hold as equalities. Suppose first that (B.12) holds as an equality so that $\underline{U}_S = 0$. Then, there exists a solution $(H_B^\star, H_S^\star)$ satisfying (B.10). Indeed, (B.10) is clear if the two additional binding constraints in the (sub)system (B.5)–(B.9) are constraints (B.8). In the alternative case when (B.9) binds, we conclude from $[\underline{q}, \bar{q}] \equiv [\max\{q_S^1, q_B^1\}, \min\{q_S^3, q_B^3\}]$ that one of $H_j^\star$ must be degenerate (supported on a singleton), so the claim follows as well. Next, suppose that (B.12) holds as a strict inequality. Then, there exists a solution $(H_B^\star, H_S^\star)$ satisfying (B.11) because additional three inequalities must be equalities in the (sub)system (B.5)–(B.9).

Now consider the second case in which $\alpha^\star > \max\{\Lambda_B, \Lambda_S\}$. Then, we must have $\underline{U}_B = \underline{U}_S = 0$ in all solutions. Thus, the linear system (B.5)–(B.9) has four equality constraints and six free variables (once $\underline{U}_S$ and $\underline{U}_B$ are fixed). By the same reasoning as above, there exists a solution $(H_B^\star, H_S^\star)$ satisfying (B.10). This finishes the proof of the claim. □

Finally, we translate our results on the cardinality of the support of $(H_B^\star, H_S^\star)$ into our mechanism characterization.

Claim 3. *If* $|\operatorname{supp}(H_B^\star)| + |\operatorname{supp}(H_S^\star)| \leq m$*, then the corresponding direct mechanism offers at most* $m - 2$ *rationing options in total.*

Proof. We consider the seller side; the argument for the buyer side is analogous. Suppose that $|\operatorname{supp}(H_S^\star)| = n$. Let $r_S^k = G_S^{-1}(q_S^k)$, for all $k = 1, \ldots, n$. Then, the corresponding optimal $X_S(r)$ is given by

$$X_S(r) = \sum_{k=1}^{n} \nu_S^k \mathbf{1}_{\{r \leq r_S^k\}}.$$

By direct inspection, $|\mathrm{Im}(X_S) \setminus \{0, 1\}| \leq n - 1$, so the conclusion follows. □

Theorem 1 follows from Claim 3 and the restrictions on $|\operatorname{supp}(H_B^\star) \cup \operatorname{supp}(H_S^\star)|$ derived in Claim 2.

B.3 Proof of Theorem 2

We show that under the assumptions of the theorem, the functions $\phi_j^{\alpha^\star}$ are strictly concave with the optimal Lagrange multiplier $\alpha^\star$. This is sufficient to prove optimality of a competitive mechanism because of Lemma 1—when the objective function is strictly concave, it coincides with its concave closure and the unique optimal distribution of quantities is degenerate, corresponding to a competitive mechanism.

As argued in the proof of Theorem 1, we must have $\alpha^\star \geq \max\{\Lambda_S, \Lambda_B\}$. Then, the derivative of the function $\phi_S^{\alpha^\star}(q)$ takes the form

$$(\phi_S^{\alpha^\star})'(q) = \Pi_S^\Lambda(G_S^{-1}(q)) - \alpha^\star J_S(G_S^{-1}(q)),$$

so it is enough to prove that

$$\Pi_S^\Lambda(r) - \alpha^\star J_S(r) \tag{B.13}$$

is strictly decreasing in r. Rewriting (B.13), we have

$$\Pi_S^\Lambda(r) - \alpha^\star J_S(r) = \Lambda_S \underbrace{\left[\frac{\int_{\underline{r}_S}^{r}[\bar{\lambda}_S(\tau) - 1]g_S(\tau)d\tau}{g_S(r)} - r\right]}_{\Delta_S(r) - r} - (\alpha^\star - \Lambda_S)J_S(r).$$

Virtual surplus $J_S(r)$ is non-decreasing, and $\alpha^\star \geq \Lambda_S$, so it is enough to prove that the first term is strictly increasing. The function $\Delta_S(r) - r$ is strictly quasi-concave by Assumption 1, so to prove strict monotonicity on the entire domain, it is enough to show that the derivative at $r = \underline{r}_S$ is negative. We have

$$\frac{d}{dr}[\Delta_S(r) - r]|_{r=\underline{r}_S} = \bar{\lambda}_S(\underline{r}_S) - 2 \leq 0,$$

where the last inequality follows from the assumption that same-side inequality is low (recall that $\Lambda_S \bar{\lambda}_S(\underline{r}_S) = \lambda_S(\underline{r}_S)$).

We now show that $\phi_B^{\alpha^\star}(q)$ is also strictly concave:

$$(\phi_B^{\alpha^\star})'(q) = \Pi_B^\Lambda(G_B^{-1}(1-q)) + \alpha^\star J_B(G_B^{-1}(1-q)),$$

so it is enough to show that

$$\Pi_B^\Lambda(r) + \alpha^\star J_B(r) \tag{B.14}$$

is increasing. Rewriting (B.14), we have

$$\Pi_B^\Lambda(r) + \alpha^\star J_B(r) = \Lambda_B\left[r - \Delta_B(r)\right] + (\alpha^\star - \Lambda_B)J_B(r).$$

Because the virtual surplus function $J_B(r)$ is non-decreasing, and $\alpha^\star \geq \Lambda_B$, by assumption, it is enough to prove that $r - \Delta_B(r)$ is increasing. Because this function is strictly quasi-convex by Assumption 1, it is enough to prove that the derivative is non-negative at the end point $r = \underline{r}_B$:

$$\frac{d}{dr}[r - \Delta_B(r)]|_{r=\bar{r}_B} = 2 - \bar{\lambda}_B(\underline{r}_B) \geq 0,$$

by the assumption that buyer-side inequality is low. Thus, we have proven that both functions $\phi_j^{\alpha^\star}$ are strictly concave.

It follows that a competitive mechanism with no rationing is optimal for both sides of the market, and the revenue (if strictly positive) is redistributed to the sellers if $\Lambda_S \geq \Lambda_B$, and to the buyers otherwise (see Theorem 1). Concavity of $\phi_j^{\alpha^\star}$ implies that the first-order condition in problem (4.6) has to hold and is sufficient for optimality. This means that the optimal volume of trade $Q^\star \in [0,\, \mu \wedge 1]$ (the maximizer of the right hand side of (4.6)) satisfies

$$\Pi_S^\Lambda(G_S^{-1}(Q^\star)) - \alpha^\star J_S(G_S^{-1}(Q^\star)) + \Pi_B^\Lambda\left(G_B^{-1}\left(1 - \frac{Q^\star}{\mu}\right)\right) + \alpha^\star J_B\left(G_B^{-1}\left(1 - \frac{Q^\star}{\mu}\right)\right) \geq 0$$
$$(= 0 \text{ when } Q^\star < \mu \wedge 1). \tag{B.15}$$

Rewriting (B.15), and noting that $p_S = G_S^{-1}(Q^\star)$ and $p_B = G_B^{-1}(1 - \frac{Q^\star}{\mu})$,

$$\Lambda_S[\Delta_S(p_S) - p_S] - (\alpha^\star - \Lambda_S)J_S(p_S) + \Lambda_B[p_B - \Delta_B(p_B)] + (\alpha^\star - \Lambda_B)J_B(p_B) \geq 0$$
$$(= 0 \text{ when } Q^\star < \mu \wedge 1). \tag{B.16}$$

Moreover, prices p_B, p_S have to satisfy $p_B \geq p_S$ and clear the market:

$$\mu(1 - G_B(p_B)) = G_S(p_S). \tag{B.17}$$

First, assume that (5.3) holds at the competitive-equilibrium price p^{CE}; we show that in this case, competitive-equilibrium is optimal. At p^{CE}, market-clearing and budget-balance hold, by construction (with $\underline{U}_S = \underline{U}_B = 0$). Therefore, we only have to prove existence of $\alpha^\star \geq \max\{\Lambda_S, \Lambda_B\}$ such that the first-order condition holds:

$$\Lambda_S[\Delta_S(p^{\mathsf{CE}}) - p^{\mathsf{CE}}] - (\alpha^\star - \Lambda_S)J_S(p^{\mathsf{CE}}) + \Lambda_B[p^{\mathsf{CE}} - \Delta_B(p^{\mathsf{CE}})] + (\alpha^\star - \Lambda_B)J_B(p^{\mathsf{CE}}) \geq 0 \tag{B.18}$$

with equality when the solution is interior (i.e., when $p^{CE} \in (\underline{r}_S, \bar{r}_B)$). Simplifying (B.18) gives:

$$\Lambda_S \underbrace{\left[\Delta_S(p^{\mathsf{CE}}) + \frac{G_S(p^{\mathsf{CE}})}{g_S(p^{\mathsf{CE}})}\right]}_{\geq 0} - \Lambda_B \underbrace{\left[\Delta_B(p^{\mathsf{CE}}) - \frac{1 - G_B(p^{\mathsf{CE}})}{g_B(p^{\mathsf{CE}})}\right]}_{\leq 0} \geq \alpha^\star \left[\frac{G_S(p^{\mathsf{CE}})}{g_S(p^{\mathsf{CE}})} + \frac{1 - G_B(p^{\mathsf{CE}})}{g_B(p^{\mathsf{CE}})}\right]$$

with equality when $p^{CE} \in (\underline{r}_S, \bar{r}_B)$. Since the left hand side is non-negative, such a solution $\alpha^\star \geq \max\{\Lambda_S, \Lambda_B\}$ exists if and only if we have an inequality at the minimal possible $\alpha^\star$, that is, $\alpha^\star = \max\{\Lambda_S, \Lambda_B\}$:

$$\Lambda_S \left[\Delta_S(p^{\mathsf{CE}}) + \frac{G_S(p^{\mathsf{CE}})}{g_S(p^{\mathsf{CE}})}\right] - \Lambda_B \left[\Delta_B(p^{\mathsf{CE}}) - \frac{1 - G_B(p^{\mathsf{CE}})}{g_B(p^{\mathsf{CE}})}\right] \geq \max\{\Lambda_S, \Lambda_B\} \left[\frac{G_S(p^{\mathsf{CE}})}{g_S(p^{\mathsf{CE}})} + \frac{1 - G_B(p^{\mathsf{CE}})}{g_B(p^{\mathsf{CE}})}\right].$$

Simplifying the preceding expression shows that it is equivalent to condition (5.3).

It remains to show what the form the solution takes when condition (5.3) fails. A competitive equilibrium cannot be optimal in this case because there does not exist $\alpha^\star$ under which the corresponding quantity maximizes the Lagrangian (4.6) in Lemma 1. Conseuqently, we have $p_B > p_S$, and, in light of Theorem 1, there will be a strictly positive lump-sum payment for the "poorer" side of the market: $\underline{U}_S > 0$ when $\Lambda_S \geq \Lambda_B$ and $\underline{U}_B > 0$ when $\Lambda_B > \Lambda_S$; this implies that we must have $\alpha^\star = \max\{\Lambda_S, \Lambda_B\}$. Subsequently, the optimal prices p_B and p_S are pinned down by market-clearing (B.17) and the first-order condition (B.16) which—assuming that an interior solution exists—becomes

$$\Lambda_S(p_B - p_S) = -\Lambda_S\Delta_S(p_S) + \Lambda_B\Delta_B(p_B) + (\Lambda_S - \Lambda_B)\frac{1 - G_B(p_B)}{g_B(p_B)},$$

when $\Lambda_S \geq \Lambda_B$, and

$$\Lambda_B(p_B - p_S) = -\Lambda_S\Delta_S(p_S) + \Lambda_B\Delta_B(p_B) + (\Lambda_B - \Lambda_S)\frac{G_S(p_S)}{g_S(p_S)},$$

otherwise. When there is no interior solution, one of the prices is equal to the bound of the support of rates of substitution, and the other price is determined by the market-clearing condition. This finishes the proof of the theorem.

B.4 Proof of Theorem 3

Consider the buyer side (we normalize $\mu = 1$ to simplify notation). We can decompose $\phi_B^{\alpha^\star}$ in the following way:

$$\phi_B^{\alpha^\star}(q) = \Lambda_B \underbrace{\int_{G_B^{-1}(1-q)}^{\bar{r}_B} [r - \Delta_B(r)]g_B(r)dr}_{\phi_B^1(q)} + (\alpha^\star - \Lambda_B) \underbrace{\int_{G_B^{-1}(1-q)}^{\bar{r}_B} J_B(r)g_B(r)dr}_{\phi_B^2(q)} + (\Lambda_B - \alpha^\star)\underline{U}_B.$$

Because the virtual surplus function is non-decreasing, the function $\phi_B^2(q)$ is concave. Consider the function ϕ_B^1. We know that $\phi_B^1(0) = 0$, and that, by Assumption 1,

$$(\phi_B^1)'(q) = G_B^{-1}(1-q) - \Delta_B(G_B^{-1}(1-q))$$

is a quasi-convex function: It is non-increasing on $[0, \hat{q}]$ and non-decreasing on $[\hat{q}, 1]$, for some $\hat{q} \in [0, 1]$. It follows that $\phi_B^1(q)$ is concave on $[0, \hat{q}]$ and convex on $[\hat{q}, 1]$.

Consider $\text{co}(\phi_B^1)(q)$. By the properties of $\phi_B^1(q)$ described above, $\text{co}(\phi_1^B)(q)$ is linear on an interval $[\tilde{q}, 1]$ for some $\tilde{q}$, and $\text{co}(\phi_B^1)(q) = \phi_B^1(q)$ for all $q \leq \tilde{q}$. We show that $\tilde{q} > 0$. Suppose not, i.e., assume that $\tilde{q} = 0$, that is, the concave closure is a linear function supported at the endpoints of the domain. Then, since ϕ_B^1 is concave in the neighborhood of 0, it must be that a linear function tangent to ϕ_B^1 at $q = 0$ lies weakly below ϕ_B^1 at $q = 1$:

$$\phi_B^1(0) + (\phi_B^1)'(0)(1-0) \leq \phi_B^1(1).$$

Rewriting the above inequality, we obtain

$$\bar{r}_B \leq \int_0^1 [G_B^{-1}(1-q) - \Delta_B(G_B^{-1}(1-q))]dq,$$

or equivalently,

$$\bar{r}_B \leq \int_{\underline{r}_B}^{\bar{r}_B} [r - \Delta_B(r)]dG_B(r),$$

which is a contradiction since

$$\int_{\underline{r}_B}^{\bar{r}_B} [r - \Delta_B(r)]dG_B(r) \leq \int_{\underline{r}_B}^{\bar{r}_B} r dG_B(r) < \bar{r}_B.$$

The contradiction proves that $\tilde{q} > 0$. Finally, notice that since $\alpha^\star \geq \Lambda_B$ in the optimal mechanism, by Lemma 1, $(\alpha^\star - \Lambda_B)\phi_B^2(q)$ is a concave function which is added to ϕ_B^1 to obtain $\phi_B^{\alpha^\star}$. Thus, the region in which $co(\phi_B^{\alpha^\star})$ is linear must be contained in the region where $co(\phi_B^1)$ is linear (this follows directly from the definition of the concave closure). Therefore, $co(\phi_B^{\alpha^\star})$ cannot be linear on $[0, \tilde{q}]$, and hence coincides with $\phi_B^{\alpha^\star}(q)$ for $q \in [0, \tilde{q}]$.

We are ready to finish the first part of the proof. If there is rationing on the buyer side, then the optimal volume of trade must lie in the region where $\phi_B^{\alpha^\star}$ lies strictly below its concave closure. It follows that $Q \geq \tilde{q} > 0$, and that $\text{supp}\{H_B^\star\} \subseteq [\tilde{q}, 1]$ (we can set $\underline{Q}_B = \tilde{q}$). This means that each corresponding price $p^i = G_B^{-1}(1-q^i)$ for $q^i \in \text{supp}\{H_B^\star\}$ satisfies $p^i < \bar{r}_B$. Thus, there is non-zero measure of buyers who trade with probability one under the optimal mechanism.

The proof of the second part of the theorem for the seller side is virtually identical and thus skipped.

B.5 Proof of Theorem 4

The first part of Theorem 4 follows from Theorem 3: If there is rationing on the buyer side, there must exist a non-zero measure of buyers that trade with probability 1—and thus it is never optimal to ration at a single price.

To prove the second part of the theorem, it is enough to prove that the function $\phi_S^\alpha(q)$ is first convex and then concave, for any $\alpha \geq \Lambda_S$. Indeed, this implies that the concave closure of $\phi_S^\alpha(q)$ is a linear function on $[0, \hat{q}]$ for some $\hat{q} > 0$, and coincides with $\phi_S^\alpha(q)$ otherwise. Thus, when there is rationing, it takes the form of a lottery between the quantities $q = 0$ and $q = \hat{q}$ which corresponds to a single price with rationing.

It suffices to show that the derivative of $\phi_S^{\alpha}(q)$ is quasi-concave. Similarly as in the proof of Theorem 3, we can decompose $\phi_S^{\alpha^\star}(q)$ as

$$\phi_S^{\alpha^\star}(q) = \Lambda_S \underbrace{\int_{\underline{r}_S}^{G_S^{-1}(q)} [\Delta_S(r) - r] g_S(r) dr}_{\phi_S^1(q)} - (\alpha^\star - \Lambda_S) \underbrace{\int_{\underline{r}_S}^{G_S^{-1}(q)} J_S(r) g_S(r) dr}_{\phi_S^2(q)} + (\Lambda_S - \alpha^\star)\underline{U}_S.$$

Then, we have

$$(\phi_S^{\alpha^\star})'(q) = \Lambda_S (\phi_S^1)'(q) - (\alpha^\star - \Lambda_S)(\phi_S^2)'(q).$$

Under assumption *(i)*, sellers receive a strictly positive lump-sum transfer and hence we must have $\alpha^\star = \Lambda_S$. At the same time we have $(\phi_S^1)'(q) = \Delta_S(G_S^{-1}(q)) - G_S^{-1}(q)$ which is quasi-concave by the regularity condition (a composition of a quasi-concave function with an increasing function is quasi-concave). Under assumption *(ii)*, $(\phi_S^1)'(q)$ is a concave function, and $(\phi_S^2)'(q) = J_S(G_S^{-1}(q))$ is a convex function. Thus, the derivative of $\phi_S^{\alpha}(q)$ is concave, and hence quasi-concave.

B.6 Proof of Theorem 5

First, we prove a property of the function ϕ_S^{α}. Importantly, with α treated as a free parameter, ϕ_S^{α} is determined by the primitive variables and does not depend on μ.

Lemma 4. *There exist $\hat{q} > 0$ and $\bar{\alpha} > \Lambda_S$ such that if $\alpha < \bar{\alpha}$, then $\phi_S^{\alpha}(q)$ is strictly convex on $[0, \hat{q}]$.*

Proof. The derivative of $\phi_S^{\alpha}(q)$ is $\Pi_S^{\Lambda}(G_S^{-1}(q)) - \alpha J_S(G_S^{-1}(q))$. Because the function G_S^{-1} is strictly increasing, it is enough to prove that $\Pi_S^{\Lambda}(r) - \alpha J_S(r)$ is strictly increasing for $r \in [\underline{r}_S, \hat{r}]$, for some $\hat{r}$ (we then set $\hat{q} = G_S(\hat{r})$). Taking a derivative again, and rearranging, yields the following sufficient condition: for $r \in [\underline{r}_S, \hat{r}]$,

$$\bar{\lambda}_S(r) > 2 + \frac{g_S'(r)}{g_S(r)} \Delta_S(r) + \frac{\alpha - \Lambda_S}{\Lambda_S} \left[2 - \frac{g_S'(r) G_S(r)}{g_S^2(r)} \right].$$

Because g_S was assumed continuously differentiable and strictly positive, including at $r = \underline{r}_S$, we can put a uniform (across r) bound $M < \infty$ on $\frac{g_S'(r)}{g_S(r)}$ and $2 - \frac{g_S'(r) G_S(r)}{g_S^2(r)}$.

B.7 Proof of Theorem 6

By Theorem 2, we know that rationing cannot be optimal for buyers when there is low buyer-side inequality, so we can assume that buyer-side inequality is high without loss of generality.

Let $\phi_B^1(q)$ and $\phi_B^2(q)$ be defined as in proof of Theorem 3, and normalize $\mu = 1$ (it plays no role in this part of the proof). Recall that $(\phi_B^1)'(q) = G_B^{-1}(1-q) - \Delta_B(G_B^{-1}(1-q))$. The function $r - \Delta_B(r)$ is strictly quasi-convex by Assumption 1. The function $G_B^{-1}(1-q)$ is strictly decreasing. A composition of strictly quasi-convex function with a strictly decreasing function is strictly quasi-convex. Therefore, $(\phi_B^1)'(q)$ is strictly decreasing on $[0, \bar{q}]$ and strictly increasing on $[\bar{q}, 1]$ for some $0 \le \bar{q} \le 1$. Moreover, $(\phi_B^1)'(0) = \bar{r}_B > 0$ and $(\phi_B^1)'(1) = \underline{r}_B = 0$. It follows that $(\phi_B^1)'(q)$ is negative whenever it is increasing, and thus $\phi_B^1(q)$ is decreasing whenever it is convex. Because $\phi_B^1(0) = 0$ and $(\phi_B^1)'(0) > 0$, it follows that $\phi_B^1(q)$ is (strictly) concave on $[0, q^\star]$, where $q^\star$ achieves the global maximum of $\mathrm{co}(\phi_1^B)(q)$ over all $q \in [0, 1]$.

We now prove that the above property of $\phi_B^1(q)$ continues to hold for $\phi_B^{\alpha^\star}(q)$.

Lemma 5. *Suppose that the function co($\phi_B^{\alpha^\star}$) has a global maximum at $Q^\star$. Then, $\phi_B^{\alpha^\star}$ is strictly concave on $[0, Q^\star]$ (and in particular equal to co($\phi_B^{\alpha^\star}$)).*

Proof. The proof differs depending on which assumption, *(i)* or *(ii)*, is satisfied.

(i) When buyers receive a strictly positive lump-sum transfer, then we must have $\alpha^\star = \Lambda_B$. It follows that $\phi_B^{\alpha^\star}(q) = \Lambda_B \phi_B^1(q) + (\Lambda_B - \alpha^\star)\underline{U}_B$, and hence $\phi_B^{\alpha^\star}$ immediately inherits the required property from ϕ_B^1.

(ii) We have $(\phi_B^2)'(q) = J_B(G_B^{-1}(1-q))$, and thus $(\phi_B^2)'(q)$ is convex by assumption. Similarly, $(\phi_B^1)'(q)$ is convex by assumption. Therefore $(\phi_B^{\alpha^\star})'(q)$ is convex, and moreover $(\phi_B^{\alpha})'(1) \le 0$. Therefore, $\phi_B^{\alpha^\star}$ has the same property as $\phi_B^1(q)$ (by the same argument). □

The proof of the first part of the theorem now follows from Lemma 5. First, notice that $\phi_S^{\alpha}(q)$ (and hence also $\mathrm{co}(\phi_S^{\alpha})(q)$) is non-increasing in q for any $\alpha \ge \Lambda_S$ under the assumptions of the theorem. This follows from $(\phi_S^{\alpha})'(0) = -\alpha \underline{r}_S \le 0$ and the proof of Theorem 2 where we showed that under the regularity condition and low seller-side inequality, $(\phi_S^{\alpha})'(q)$ is strictly decreasing. This implies that Q, the maximizer of the Lagrangian (4.6), must be lower than $Q^\star$—the maximizer of co($\phi_B^{\alpha^\star}$) from Lemma 5.

But then, by Lemma 5, $\phi_B^{\alpha^\star}(q)$ is strictly concave on $[0, Q^\star]$ and coincides with its concave closure at $q = Q$. Thus, there cannot be rationing on the buyer side.

We prove the second part of Theorem 6 by showing that the Lagrange multiplier can be taken to be $\alpha^\star = \Lambda_B$ in this case (we no longer assume that $\mu = 1$ as this is not without loss of generality for this part of the proof). From the first part of the proof, we know that with $\alpha = \Lambda_B$, the function $\phi_B^\alpha(q)$ is first concave and then convex, and that it achieves its global maximum on the part of the domain where it is concave. Because seller-side inequality is low (so that $\phi_S^\alpha(q)$ is decreasing and concave), it is sufficient to prove that the first-order condition is satisfied (see the proof of Theorem 2),

$$\Lambda_S[\Delta_S(p_S) - p_S] - (\Lambda_B - \Lambda_S)J_S(p_S) + \Lambda_B[p_B - \Delta_B(p_B)] = 0, \tag{B.19}$$

the market clears,

$$\mu(1 - G_B(p_B)) = G_S(p_S), \tag{B.20}$$

and budget-balance is maintained: Because we aim to prove that a competitive mechanism is optimal for both sides, and $\alpha^\star = \Lambda_B$ implies that $\underline{U}_B$ can be an arbitrary positive number, it is enough if we prove that

$$p_B \geq p_S. \tag{B.21}$$

Thus, we seek to prove existence of a solution $(p_B^\star, p_S^\star)$ to the system (B.19) - (B.20) which additionally satisfies (B.21). First, notice that (B.20) can be equivalently written as

$$p_S = \psi(p_B) \equiv G_S^{-1}(\mu(1 - G_B(p_B))),\, p_B \in [\underline{p}_B, \bar{r}_B],$$

where $\underline{p}_B = G_B^{-1}(\max(0, 1 - \frac{1}{\mu}))$ (when $\mu > 1$, there cannot exist a solution in which $p_B < \underline{p}_B$). Therefore, we can write a single equation for $p \in [\underline{p}_B, \bar{r}_B]$ as

$$\Phi(p) \equiv \Lambda_S[\Delta_S(\psi(p)) - \psi(p)] - (\Lambda_B - \Lambda_S)J_S(\psi(p)) + \Lambda_B[p - \Delta_B(p)] = 0.$$

The function $\Psi(p)$ is continuous in p, and we have

$$\Phi(\bar{r}_B) = \Lambda_S[\Delta_S(\underline{r}_S) - \underline{r}_S] - (\Lambda_B - \Lambda_S)J_S(\underline{r}_S) + \Lambda_B[\bar{r}_B - \Delta_B(\bar{r}_B)] = -\Lambda_B\underline{r}_S + \Lambda_B\bar{r}_B > 0.$$

There are two cases to consider. When $\mu \leq 1$, we have $\underline{p}_B = \underline{r}_B$, $\psi(\underline{p}_B) = G_S^{-1}(\mu)$, and thus

$$\Phi(\underline{p}_B) \leq \Lambda_B \underline{r}_B = 0,$$

by assumption. In the opposite case $\mu > 1$, we have $\underline{p}_B = G_B^{-1}(1 - \frac{1}{\mu})$, $\psi(\underline{p}_B) = \bar{r}_S$, and thus

$$\Phi(\underline{p}_B) \leq -\Lambda_B \bar{r}_S + \Lambda_B G_B^{-1}\left(1 - \frac{1}{\mu}\right) \leq -\Lambda_B \bar{r}_S + \Lambda_B \bar{r}_B \leq 0,$$

using the assumption that $\bar{r}_S \geq \bar{r}_B$. In both cases we conclude that $\Phi(\underline{p}_B) \leq 0$. Because the function Φ changes sign, there exists $p_B^\star$ such that $\Phi(p_B^\star) = 0$, and then $p_S^\star = \psi(p_B^\star)$ is well defined as well.

It remains to prove that this solution $(p_B^\star, p_S^\star)$ satisfies (B.21). Rewrite the first-order condition as

$$p_B - p_S = \Delta_B(p_B) - \frac{\Lambda_S}{\Lambda_B}\Delta_S(p_S) + \frac{\Lambda_B - \Lambda_S}{\Lambda_B}\frac{G_S(p_S)}{g_S(p_S)}.$$

Under assumption *(b)*, there is no seller-side inequality and thus $\Delta_S(p_S) \equiv 0$. Because $\Delta_B(p_B) > 0$ and $\Lambda_B \geq \Lambda_S$, we conclude that $p_B > p_S$. Under assumption *(a)*, we have

$$p_B - p_S \geq \Delta_B(p_B) - \frac{1}{2}\Delta_S(p_S) + \frac{1}{2}\frac{G_S(p_S)}{g_S(p_S)} \geq \frac{1}{2}\frac{\int_{\underline{r}_S}^{p_S}[2 - \bar{\lambda}_S(r)]dG_S(r)}{g_S(p_S)} > 0,$$

because seller-side inequality is low ($\bar{\lambda}_S(r) \leq 2$ for all r).

This finishes the proof: The fact that $p_B > p_S$ implies that there is a strictly positive revenue in the mechanism, and the fact that $\alpha^\star = \Lambda_B$ implies that the revenue in the optimal mechanism is redistributed as a lump-sum payment to buyers.

B.8 Proof of Theorem 7

Under the assumptions of Theorem 7, we have that $\underline{r}_B > \bar{r}_S$; thus, any feasible (in particular optimal) mechanism must feature a strictly positive lump-sum transfer and $\alpha^\star = \Lambda_B \geq \Lambda_S$ (by the proof of Theorem 1). We prove that $\mu \text{co}(\phi_B^{\alpha^\star})(Q/\mu) + \text{co}(\phi_S^{\alpha^\star})(Q)$ is increasing in Q. Set $M = 1/g_B(\underline{r}_B) + 1/g_S(\bar{r}_S)$—a finite constant.

When $\alpha^\star = \Lambda_B$, we have

$$(\phi_B^{\alpha^\star})'(q) = \Lambda_B \left[G_B^{-1}(1-q) - \Delta_B(G_B^{-1}(1-q))\right] \geq \Lambda_B \left[G_B^{-1}(1-q) - \frac{1-G_B(G_B^{-1}(1-q))}{g_B(G_B^{-1}(1-q))}\right]$$
$$= \Lambda_B J_B(G_B^{-1}(1-q)) \geq \Lambda_B J_B(\underline{r}_B),$$

where the last inequality follows from the fact that virtual surplus is monotone by assumption. Hence, we have

$$\inf_q \left\{\frac{d}{dq}\left[\mu \mathrm{co}(\phi_B^{\alpha^\star})(q/\mu)\right]\right\} = \inf_q \left\{\mathrm{co}(\phi_B^{\alpha^\star})'(q/\mu)\right\} \geq \inf_q \left\{(\phi_B^{\alpha^\star})'(q/\mu)\right\} \geq \Lambda_B J_B(\underline{r}_B),$$

using the fact that the derivative of the concave closure of a function is lower bounded by the infimum of the derivatives of that function.

Similarly, on the seller side we have

$$(\phi_S^{\alpha^\star})'(q) = \Lambda_S \left[\Delta_S(G_S^{-1}(q) - G_S^{-1}(q)\right] - (\Lambda_B - \Lambda_S) J_S(G_S^{-1}(q)) \geq$$
$$\Lambda_S \left[-\frac{G_S(G_S^{-1}(q))}{g_S(G_S^{-1}(q))} - G_S^{-1}(q)\right] - (\Lambda_B - \Lambda_S) J_S(G_S^{-1}(q)) = -\Lambda_B J_S(G_S^{-1}(q)) \geq -\Lambda_B J_S(\bar{r}_S),$$

using the assumption that virtual cost is monotone. Therefore,

$$\inf_q \left\{\mathrm{co}(\phi_S^{\alpha^\star})'(q)\right\} \geq \inf_q \left\{(\phi_S^{\alpha^\star})'(q)\right\} \geq -\Lambda_B J_S(\bar{r}_S).$$

The obtained inequalities imply that the derivative of $\mu \mathrm{co}(\phi_B^{\alpha^\star})(Q/\mu) + \mathrm{co}(\phi_S^{\alpha^\star})(Q)$ is lower bounded by

$$\Lambda_B \left[J_B(\underline{r}_B) - J_S(\bar{r}_S)\right] = \Lambda_B \left[\underline{r}_B - \bar{r}_S - \underbrace{\left(\frac{1}{g_B(\underline{r}_B)} + \frac{1}{g_S(\bar{r}_S)}\right)}_{M}\right]$$

which is non-negative by assumption of the theorem. Because the Lagrangian $\mu \mathrm{co}(\phi_B^{\alpha^\star})(Q/\mu) +$ $\mathrm{co}(\phi_S^{\alpha^\star})(Q)$ is non-decreasing, the optimal volume of trade is equal to the maximal feasible quantity: $Q = \min\{\mu, 1\}$. Assume that $\mu > 1$ so that $Q = 1$.

To finish the proof, recall from the proof of Theorem 3 that, when $\alpha^\star = \Lambda_B$ and buyer-side inequality is high, the function $\phi_B^{\alpha^\star}$ lies strictly below its concave closure when the fraction of buyers trading is sufficiently close to 1.[39] Because the optimal volume of trade is 1 and the mass of buyers is μ, when $\mu \in (1,\, 1+\epsilon)$, the fraction of buyers trading in the optimal mechanism is arbitrarily close to 1 for small ϵ. Thus, there exists $\epsilon > 0$ such that the optimal mechanism rations the buyers whenever $\mu \in (1,\, 1+\epsilon)$ (rationing is equivalent to $\phi_B^{\alpha^\star}$ lying below its concave closure at the optimal volume of trade).

B.9 Proofs of results in Section 3

Finally, we explain how the results stated in Section 3 follow from the general results stated in Sections 4 and 5.

First, note that while the one-sided problems considered in Section 3 are formally different from the two-sided problem studied in Sections 4 and 5, the techniques extend immediately to this case because most of our analysis looked at the two sides of the market separately. In particular, optimality of rationing on side j depends solely on the properties of the function $\phi_j^{\alpha^\star}$. This is still the case in the one-sided problem. The only differences are that *(i)* the budget constraint has an exogenous revenue level R, and *(ii)* Q is fixed rather than determined endogenously. Thus, optimality of rationing depends on whether or not the function $\phi_j^{\alpha^\star}$ lies below its concave closure at the fixed quantity Q.

Next, we note that under the assumption of uniform distribution, all of the functions $G_B^{-1}(q) - \Delta_B(G_B^{-1}(q))$, $J_B(G_B^{-1}(q)$, $G_S^{-1}(q) - \Delta_S(G_S^{-1}(q))$, and $J_S(G_S^{-1}(q))$ are convex. By inspection of the proof of Theorem 3, this implies that regardless of the Lagrange multiplier α, the function $\phi_B^{\alpha}(q)$ is first concave and then convex, and the function $\phi_S^{\alpha}(q)$ is first convex and then concave. Consequently, we observe that there exists q_B^{α} such that rationing on the buyer side is optimal if and only if $Q \in (q_B^{\alpha},\, 1)$ (with μ normalized to 1). Similarly, there exists q_S^{α} such that rationing on the seller side is optimal if and only if $Q \in (0,\, q_S^{\alpha})$.

Proof of Proposition 1

When seller-side inequality is low, the function $\phi_S^{\alpha}(q)$ is strictly concave and thus a competitive mechanism is optimal.

[39] In the proof of Theorem 3, we normalized $\mu = 1$; thus, q close to 1 in the proof of Theorem 3 should be interpreted as q close enough to μ when μ is arbitrary.

When seller-side inequality is high, rationing is optimal if and only if $Q \in (0, q_S^\alpha)$ for q_S^α defined above. Moreover, by Theorem 4, whenever it is optimal to ration, it is optimal to ration at a single price. We can define $Q(R)$ as q_S^α with $\alpha = \alpha_R^\star$ being the optimal Lagrange multiplier on the budget constraint with revenue target R. Then, to establish Proposition 1, it only remains to show the three properties of the function $Q(R)$:

1. $Q(R)$ is strictly positive for high enough R; indeed, when R is high enough, sellers must receive a strictly positive lump-sum transfer in the optimal mechanism. But then, we must have $\alpha_R^\star = \Lambda_S$, and thus $q_S^\alpha > 0$, by the proof of Theorem 3.

2. $Q(R) < 1$ for all R; this follows directly from Theorem 3.

3. $Q(R)$ is increasing; this follows from two claims: First, the optimal Lagrange multiplier $\alpha_R^\star$ is decreasing in the revenue level R (a higher R corresponds to an easier-to-satisfy constraint, so the corresponding Lagrange multiplier must be lower);[40] Second, $\phi_S^{\alpha_1} - \phi_S^{\alpha_2}$ is a concave function when $\alpha_1 \geq \alpha_2$; thus, the set of points at which $\phi_S^{\alpha_1}$ lies below its concave closure is contained in the set of points at which $\phi_S^{\alpha_2}$ lies below its concave closure. It follows that q_S^α is decreasing in α. Putting the two preceding observations together, we conclude that $Q(R)$ is increasing.

Proof of Proposition 2

Differentiating the designer's objective function over p_B yields

$$Q\left[\underbrace{\frac{g_B(p_B)}{(1-G_B(p_B))^2}\int_{p_B}^{\bar{r}_B}\lambda_B(r)(r-p_B)dG_B(r)}_{\geq 0}+\Lambda_B-\frac{1}{1-G_B(p_B)}\int_{p_B}^{\bar{r}_B}\lambda_B(r)dG_B(r)\right]$$
$$\geq Q\left[\Lambda_B-\frac{\int_{p_B}^{\bar{r}_B}\lambda_B(r)dG_B(r)}{1-G_B(p_B)}\right]=Q\left[\mathbb{E}^B[\lambda_B(r)]-\mathbb{E}^B[\lambda_B(r)|\,r\geq p_B]\right]\geq 0, \quad \text{(B.22)}$$

[40] Formally, this claim follows from analyzing the dual problem: The Lagrange multiplier is equal to the optimal dual variable in the dual problem; a lower constant R implies that the dual variable in the dual objective function is multiplied by a smaller positive scalar; thus the optimal $\alpha_R^\star$ cannot increase.

where the last inequality follows from the fact that $\lambda_B(r)$ is non-increasing (note that this inequality corresponds to the comparison of forces *(ii)* and *(iii)* described in the discussion of Proposition 2). This shows that the objective function of the designer is non-decreasing in the choice variable; thus, it is optimal to set p_B to be equal to its upper bound $G_B^{-1}(1-Q) = p_B^{\mathsf{C}}$.

Proofs of Propositions 3–7

The argument for Proposition 3 is fully analogous to the proof of Proposition 1, and thus skipped. Proposition 4 is a special case of Theorem 2. Proposition 5 is a special case of Theorem 5; the conclusion that rationing happens at a single price follows from Theorem 4. Proposition 6 is a special case of Theorem 6. Finally, Proposition 7 is a special case of Theorem 7. Note that the constant M in the proof of Theorem 7 is given by $M = 1/g_B(\underline{r}_B) + 1/g_S(\bar{r}_S)$; specializing to the case of uniform distribution gives us the condition assumed in Proposition 7.