ONLINE APPENDIX: ADDITIONAL PROOFS, DISCUSSION AND EXTENSIONS FOR TARGETING INTERVENTIONS IN NETWORKS

Throughout the online appendix, we refer often to sections, results, and equations in the main text and its appendix using the numbering established there. The numbers of sections, results, and equations in this online appendix are all prefixed by OA to distinguish them.

A note on notation. Throughout this appendix, when i appears as the index of summation without further specification, the summation runs over the set $\mathcal{N}$ of nodes. When ℓ appears as the index of summation, the summation runs of the set $\ell = 1, \ldots, n$.

OA1. Additional Proofs

Proof of Proposition 3. Using expression (8), we can write the dependence of $\mathbb{E}\left[W(\boldsymbol{b};\boldsymbol{G})\right]$ on intervention $\mathcal{B}_{\boldsymbol{y}}$ as follows:

$$\mathbb{E}\left[W(\boldsymbol{b};\boldsymbol{G})\right] = w\sum_{\ell}\alpha_{\ell}\left(\left\{\mathbb{E}[\hat{\underline{b}}_{\ell}] + \underline{y}_{\ell}\right\}^{2} + \mathrm{Var}[\underline{b}_{\ell}]\right).$$

Choosing $\boldsymbol{y}$ to maximize this is identical to the problem analyzed in the deterministic setting in the proof of Theorem 1. Thus, defining $x_{\ell} = \underline{y}_{\ell}/\bar{b}_{\ell}$, with $\bar{b}_{\ell} = \mathbb{E}[\hat{\underline{B}}_{\ell}]$, it satisfies the same conditions at the optimum as those derived in Theorem 1. $\square$

Proof of Proposition 4. Given Assumption 5, without loss of generality we can normalize $\bar{\boldsymbol{b}} = \boldsymbol{0}$. Using expression (8) and normalization, we obtain that if the optimal solution is $\mathcal{B}^{*}$ the expected welfare obtained is

$$\mathbb{E}\left[W(\boldsymbol{b}^{*};\boldsymbol{G})\right] = w\sum_{\ell}\alpha_{\ell}\mathrm{Var}(\underline{b}_{\ell}^{*}).$$

Note that the random variable $\underline{\mathcal{B}}^{*}$ can be written as $\boldsymbol{U}^{\mathsf{T}}\mathcal{B}^{*}$, and so the variance–covariance matrix of the random variable $\underline{\mathcal{B}}^{*}$ is $\boldsymbol{\Sigma}_{\underline{\mathcal{B}}^{*}} = \boldsymbol{U}^{\mathsf{T}}\boldsymbol{\Sigma}_{\mathcal{B}^{*}}\boldsymbol{U}$, where recall that $\boldsymbol{\Sigma}_{\mathcal{B}^{*}}$ is the variance–covariance matrix of the random variable $\mathcal{B}^{*}$.

We consider the case of $w > 0$ and $\beta > 0$; the proof of the other cases is analogous and therefore omitted. The expected welfare is a weighted sum of the variances of the principal components, $\mathrm{Var}(\underline{b}_{\ell}^{*}) = \mathrm{Var}(\boldsymbol{u}^{\ell}(\boldsymbol{G})\cdot\boldsymbol{b}^{*})$, and the weight α_{ℓ} on the variance of principal component ℓ of $\boldsymbol{G}$ is an increasing function of its eigenvalue λ_{ℓ}, because $\beta > 0$.

Suppose that the claim in Proposition is violated, that is, there exists a ℓ, ℓ' such that $\ell < \ell'$ and $\mathrm{Var}(\underline{b}_{\ell}^{*}) < \mathrm{Var}(\underline{b}_{\ell'}^{*})$. We construct an alternative intervention that has the same cost and does strictly better. Take the permutation matrix (and therefore an orthogonal matrix) $\boldsymbol{P}$ such that $P_{kk} = 1$ for all $k \notin \{\ell, \ell'\}$ and $P_{\ell\ell'} = P_{\ell'\ell} = 1$. Define $\mathcal{B}^{**} = \boldsymbol{O}\mathcal{B}^{*}$ with $\boldsymbol{O} = \boldsymbol{U}\boldsymbol{P}\boldsymbol{U}^{\mathsf{T}}$. Clearly, $\boldsymbol{O}$ is orthogonal, as $\boldsymbol{U}$ and $\boldsymbol{P}$ are both orthogonal. Hence, by Assumption 5, $K(\mathcal{B}^{*}) = K(\mathcal{B}^{**})$. Furthermore, the matrix

$$\boldsymbol{\Sigma}_{\underline{\mathcal{B}}^{**}} = \boldsymbol{P}\boldsymbol{\Sigma}_{\underline{\mathcal{B}}^{*}}\boldsymbol{P}^{\mathsf{T}}$$

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and so $\mathrm{Var}(\underline{b}_k^{**}) = \mathrm{Var}(\underline{b}_k^{*})$ for all $k \notin \{\ell, \ell'\}$ and $\mathrm{Var}(\underline{b}_\ell^{**}) = \mathrm{Var}(\underline{b}_{\ell'}^{*}) > \mathrm{Var}(\underline{b}_{\ell'}^{**}) = \mathrm{Var}(\underline{b}_\ell^{*})$. Since $\alpha_\ell > \alpha_{\ell'}$ intervention $\mathcal{B}^{**}$ does strictly better than $\mathcal{B}^*$, a contradiction to our initial hypothesis that $\mathcal{B}^*$ was optimal. $\square$

OA2. Discussion

We discuss the relation of principal components of the matrix of interactions with other related networks statistics (Session OA2.1). We then provide a different economic example, which complements those in our main text, inspired by beauty context games (Session OA2.2).

OA2.1. **Principal components and other network measures.** *First principal component and eigenvector centrality:* For ease of exposition, let the network be connected, that is, let $\boldsymbol{G}$ be irreducible. By the Perron–Frobenius Theorem, $\boldsymbol{u}^1(\boldsymbol{G})$ is entry-wise positive; indeed, this vector is the Perron vector of the matrix, also known as the vector of individuals' eigenvector centralities. Thus, our results of Section 4 imply that, under strategic complementarities, interventions that aim to maximize the aggregate utility should change individuals' incentives in proportion to their eigenvector centralities.

It is worth comparing this result with results that highlight the importance of Bonacich centrality. Under strategic complements, equilibrium actions are proportional to the individuals' Bonacich centralities in the network (Ballester et al., 2006).[1] Within the Ballester et al. (2006) framework, it can easily be verified that if the objective of the planner is linear in the sum of actions, then under a quadratic cost function the planner will target individuals in proportion to their Bonacich centralities (see also Demange (2017)). Bonacich centrality converges to eigenvector centrality as the spectral radius of $\beta\boldsymbol{G}$ tends to 1; otherwise the two vectors can be quite different (see, for example, Calvó-Armengol et al. (2015) or Golub and Lever (2010)).

The substantive point is that the objective of our planner when solving the intervention problem (IT) is to maximize the aggregate equilibrium *utility*, not the sum of actions, and that explains the difference in the targeting strategy. Indeed, our planner's objective (under Property A) can be written as follows (introducing a different constant factor for convenience):

$$\sum_i u_i \propto \frac{1}{n}\sum_i a_i^2 \quad = \quad \bar{a}^2 + \sigma_{\boldsymbol{a}}^2,$$

where $\sigma_{\boldsymbol{a}}^2$ is the variance of the action profile and $\overline{a}$ is the mean action. Thus, our planner cares about the sum of actions and also their diversity, simply as a mathematical consequence of her objective. This explains the reason why her policies differ from those that would be in effect if just the mean action were the focus. To reiterate this point, we finally note that if we consider problem (IT) but we assume that the cost of intervention is linear, that is, $K(\boldsymbol{b}, \hat{\boldsymbol{b}}) = \sum_i |b_i - \hat{b}_i|$, then the optimal intervention will target only one individual (see the discussion in Online Appendix Section OA3.3); note that the targeted individual is not necessarily the individual with the highest Bonacich centrality.

Last principal component: We have shown that in games with strategic substitutes, for large budgets interventions that aim to maximize the aggregate utility target individuals in proportion to the eigenvector of $\boldsymbol{G}$ associated to the smallest eigenvalue of $\boldsymbol{G}$, the last principal component.

There is a connection between this result and the work of Bramoullé et al. (2014). Bramoullé et al. (2014) study the set of equilibria of a network game with linear best replies and strategic

[1] For a different economic context in which eigenvector centrality reflects equilibrium outcomes, see also Elliott and Golub (2018).

substitutes. They observe that such a game is a potential game, and they derive the potential function explicitly. From this, they can deduce that the smallest eigenvalue of $\boldsymbol{G}$ is crucial for whether the equilibrium is unique, and it is also useful for analyzing the stability of a particular equilibrium.[2] The basic intuition is that the magnitude of the smallest eigenvalue determines how small changes in individuals' actions propagate, via strategic substitutes, in the network. When these amplifications are strong, multiple equilibria can emerge. Relatedly, when these amplifications are strong around an equilibrium, that equilibrium will be unstable.

Our study of the strategic substitutes case is driven by different questions, and delivers different sorts of characterizations. We assume that there is a stable equilibrium which is unique at least locally, and then we characterize optimal interventions in terms of the eigenvectors of $\boldsymbol{G}$. In general, all the eigenvectors – not just the one associated to the smallest eigenvalue – can matter. Interventions will focus *more* on the eigenvectors with smaller eigenvalues. When the budget is sufficiently large, the intervention will (in the setting of Section 4) focus on only the smallest-eigenvalue eigenvector. As discussed in Section 4, the network determinants of whether targeting is simple can be quite subtle. To the best of our knowledge, these considerations are all new in the study of network games.

Nevertheless, at an intuitive level there are important points of contact between our intuitions and those of Bramoullé et al. (2014). In our context, as discussed earlier, our planner likes to move the incentives of adjacent individuals in opposite directions. The eigenvector associated to the smallest eigenvalue emerges as the one identifying the best way to do this at a given cost, and the eigenvalue itself measures how intensely the strategic effects amplify. This "amplification" property involves forces similar to those that make the smallest eigenvalue important to stability and uniqueness in Bramoullé et al. (2014).

Spectral approaches to variance control: Acemoglu et al. (2016) give a general analysis of which network statistics matter for volatility of network equilibria. Baqaee and Farhi (2017) develop a rich macroeconomic analysis relating network measures to aggregate volatility. Though both papers note the importance of eigenvector centrality in (their analogues of) the case of strategic complements, their main focus is on how the *curvature* of best responses changes the volatility of an aggregate outcome, and which "second order" (curvature-related) network statistics are important. We use the principal components of the network to understand which first-order shocks are most amplified, and how this depends on the nature of strategic interactions.

OA2.2. **Beauty contest with local interactions.** This example is inspired by Morris and Shin (2002) and Angeletos and Pavan (2007). Individuals trade off the returns from effort against the costs, as in the first example, but also care about coordinating with others. These considerations are captured in the following payoff:

$$U_i(\boldsymbol{a}, \boldsymbol{G}) = a_i \left(\tilde{b}_i + \tilde{\beta} \sum_j g_{ij} a_j \right) - \frac{1}{2} a_i^2 - \frac{\gamma}{2} \sum_j g_{ij} [a_j - a_i]^2,$$

where we assume that $\tilde{\beta} > 0$ and $\gamma > 0$ and that $\sum_j g_{ij} = 1$ for all i, so the total interaction is the same for each individual. This formulation also relates to the theory of teams and organizational economics (see, for example, Dessein et al. (2016), Marschak and Radner

[2] For stability of equilibrium, what is relevant is the magnitude of the smallest eigenvalue of an appropriately defined subgraph of $\boldsymbol{G}$.

(1972), and Calvó-Armengol et al. (2015)). We may interpret individuals as managers in different divisions within an organization. Each manager selects the action that maximizes the output of the division, given by the first term, but the manager also cares about coordinating with other divisions' actions.[3] This is a game of strategic complements; moreover, an increase in j's action has a positive effect on individual i's utility if and only if $a_j < a_i$. It can be verified that the first-order condition for individual i is given by

$$a_i = \frac{\tilde{b}_i}{1+\gamma} + \frac{\tilde{b}_i + \gamma}{1+\gamma} \sum g_{ij} a_j.$$

By defining $\beta = \frac{\tilde{\beta}+\gamma}{1+\gamma}$ and $\boldsymbol{b} = \frac{1}{1+\gamma}\tilde{\boldsymbol{b}}$, we obtain a best-response structure exactly as in condition (2). Moreover, the aggregate equilibrium utility is $W(\boldsymbol{b}, \boldsymbol{g}) = \frac{1}{2}\left(\boldsymbol{a}^*\right)^\mathsf{T} \boldsymbol{a}^*$. Hence, this game satisfies Property A.

[3]A similar analysis can be adapted to a standard (local) beauty contest game in which $U_i(\boldsymbol{a}, \boldsymbol{G}) = -(a_i - \tilde{b}_i)^2 - \gamma \sum_j g_{ij}[a_j - a_i]^2$. Here, we focus on a modification of the standard beauty contest game that makes the mapping to our formulation easier to present.

OA3. Extensions

We now extend our basic model to study settings where (a) Property A is not satisfied (Session OA3.1), (b) the matrix $\boldsymbol{G}$ is non-symmetric (Session OA3.2), (c) the exact quadratic cost specification does not hold (Session OA3.3), and (d) the interventions occur via monetary incentives for activity (Session OA3.4).

OA3.1. **General non-strategic externalities.** Section 4 characterizes optimal interventions for network games that satisfy Property A. We now relax this assumption. Recall that player i's utility for action profile $\boldsymbol{a}$ is

$$U_i(\boldsymbol{a},\boldsymbol{G}) = \hat{U}_i(\boldsymbol{a},\boldsymbol{G}) + P_i(\boldsymbol{a}_{-i},\boldsymbol{G},\boldsymbol{b}),$$

where $\hat{U}_i(\boldsymbol{a},\boldsymbol{G}) = a_i(b_i + \sum_j g_{ij}a_j) - \frac{1}{2}a_i^2$.

At an equilibrium $\boldsymbol{a}^*$, it can be checked that $\sum_i \hat{U}_i(\boldsymbol{a}^*,\boldsymbol{G}) \propto (\boldsymbol{a}^*)^{\mathsf{T}}\boldsymbol{a}^*$. Therefore, a sufficient condition for Property A to be satisfied is that $\sum_i P_i(\boldsymbol{a}^*_{-i},\boldsymbol{G},\boldsymbol{b})$ is also proportional to $(\boldsymbol{a}^*)^{\mathsf{T}}\boldsymbol{a}^*$. Examples 1 and 2, as well as the example presented in Section OA2.2, satisfy this property. However, as the next example shows, there are natural environments in which it is violated.

Example **OA1** (Social interaction and peer effects)**.** Individual decisions on smoking and alcohol consumption are susceptible to peer effects (see Jackson et al. (2017) for references to the extensive literature on this subject). For example, an increase in smoking among an adolescent's friends increases her incentives to smoke and, at the same time, has negative effects on her welfare. These considerations are reflected in the following payoff function:

$$U_i(\boldsymbol{a},\boldsymbol{G}) = \hat{U}_i(\boldsymbol{a},\boldsymbol{G}) - \gamma\sum_{j\neq i} a_j,$$

where $\beta > 0$ and γ is positive and sufficienctly large. It can be checked that the aggregate equilibrium welfare is:

$$W(\boldsymbol{b},\boldsymbol{G}) = \frac{1}{2}(\boldsymbol{a}^*)^{\mathsf{T}}\boldsymbol{a}^* - n\gamma\sum_i a_i^*, \tag{OA-1}$$

with $\boldsymbol{a}^*$ given by expression (3).[4]

To extend the analysis beyond Property A, we allow the non-strategic externality term $P_i(\boldsymbol{a}_{-i},\boldsymbol{G},\boldsymbol{b})$ to take a form that allows for flexible externalities within the linear–quadratic family:[5]

$$P_i(\boldsymbol{a}_{-i},\boldsymbol{G}) = m_1\sum_j g_{ij}a_j + m_2\sum_j g_{ij}a_j^2 + m_3\sum_{j\neq i}a_j + m_4\left(\sum_{j\neq i}a_j\right)^2 + m_5\sum_{j\neq i}a_j^2.$$

We also make the following assumption on the matrix $\boldsymbol{G}$:

Assumption OA1. The total interaction is constant across individuals, that is, $\sum_j g_{ij} = 1$ for all $i \in \mathcal{N}$.

[4]In this specification the last (externality) term is a global term. We can easily accommodate local negative externalities by replacing that term with $\sum_j g_{ij}a_j$.

[5]We can also accommodate externalities that depend directly on the b_i, but we omit this for brevity.

Using equation (3) and Assumption OA1, we can rewrite the expression for the aggregate equilibrium utility as follows:

$$W(\boldsymbol{b},\boldsymbol{G}) = w_1 \left(\boldsymbol{a}^*\right)^{\mathsf{T}} \boldsymbol{a}^* + \frac{w_2}{n}\left(\sum_i a_i^*\right)^2 + \frac{w_3}{\sqrt{n}}\sum_i a_i^*,$$

where $w_1 = 1 + m_2 + m_5 + (n-1)m_4$, $w_2 = nm_5(n-2)$, and $w_3 = \sqrt{n}[m_1 + (n-1)m_3]$.

Observe that Property A clearly holds when $w_2 = w_3 = 0$. On the other hand, if (say) $w_1 = w_2 = 0$, then the planner's objective is to maximize the sum of the equilibrium actions, which is a fairly different type of objective. A characterization of the optimal intervention when the planner's objective is to maximize the sum of the equilibrium actions can be found in Corollary OA1 below. Under Assumption OA1, the sum of the equilibrium actions is proportional to the sum of the standalone marginal returns. Because $\boldsymbol{u}^1$ is proportional to the all-ones vector $\mathbf{1}$, this sum in turn is equal to $\underline{b}_1$.

Together, these facts allow us to extend our earlier analysis to the case of general w_2 and w_3. First, we can still express the objective function simply in terms of the singular value decomposition; the only difference is that now $\underline{b}_1$ will enter both in a quadratic term and in a linear term. In view of this, we first solve the problem (exactly analogously to the previous solution) for a given value of $\underline{b}_1$, and then we optimize over $\underline{b}_1$.

We maintain Assumption 1 and Assumption 2. Recall that player i's utility for action profile $\boldsymbol{a}$ is

$$U_i(\boldsymbol{a},\boldsymbol{G}) = \hat{U}_i(\boldsymbol{a},\boldsymbol{G}) + P_i(\boldsymbol{a}_{-i},\boldsymbol{G},\boldsymbol{b})$$

where $\hat{U}_i(\boldsymbol{a},\boldsymbol{G}) = a_i(b_i + \sum_j g_{ij}a_j) - \frac{1}{2}a_i^2$ and $P_i(\boldsymbol{a}_{-i},\boldsymbol{G},\boldsymbol{b})$ is a non-strategic externality term that takes the following form:

$$P_i(\boldsymbol{a}_{-i},\boldsymbol{G}) = m_1\sum_j g_{ij}a_j + m_2\sum_j g_{ij}a_j^2 + m_3\sum_{j\neq i} a_j + m_4\left(\sum_{j\neq i} a_j\right)^2 + m_5\sum_{j\neq i} a_j^2.$$

Here we have taken local and global externality terms given by second-order polynomials in actions. (We could also accommodate externalities that depend directly on the b_i in the same sort of way, as will become clear in the proof, but we omit this for brevity.)

The implication of Assumption OA1 for our analysis is summarized next.

Lemma **OA1.** Assumption OA1 implies that:

1. for any $a \in \mathbb{R}^n$, $\sum_i\sum_j g_{ij}a_j = \sum_i a_i$ and $\sum_i\sum_j g_{ij}a_j^2 = \sum_i a_i^2$
2. $\lambda_1(\boldsymbol{G}) = 1$ and $u_i^1(\boldsymbol{G}) = \sqrt{n}$ for all i
3. $\sum_i a_i^* = \frac{1}{1-\beta}\sum b_i = \frac{\sqrt{n}}{1-\beta}\underline{b}_1 = \sqrt{n\alpha_1}\underline{b}_1$, where $\boldsymbol{a}^*$ is equilibrium action profile.[6]

The proof of Lemma OA1 is immediate. Using part 1 of Lemma OA1, and that individuals play an equilibrium (actions satisfy expression (3)), we obtain:

$$W(\boldsymbol{b},\boldsymbol{G}) = w_1 \left(\boldsymbol{a}^*\right)^{\mathsf{T}} \boldsymbol{a}^* + \frac{w_2}{n}\left(\sum_i a_i^*\right)^2 + \frac{w_3}{\sqrt{n}}\sum_i a_i^*,$$

with:

$$\begin{aligned} w_1 &= 1 + m_2 + m_5 + (n-1)m_4 \\ w_2 &= nm_5(n-2) \end{aligned}$$

[6]The last equality follows because $\alpha_1 = 1/(1-\beta\lambda_1)^2$, and assumption OA1 implies that $\lambda_1 = 1$.

$$w_3 = \sqrt{n}[m_1 + (n-1)m_3].$$

Using the decomposition $\boldsymbol{G} = \boldsymbol{U\Lambda U}^{\mathsf{T}}$, together with part 2 and part 3 of Lemma OA1, we obtain:

$$W(\underline{\boldsymbol{b}}, \boldsymbol{G}) = w_1 \underline{\boldsymbol{a}}^{*\mathsf{T}} \underline{\boldsymbol{a}}^* + w_2 \alpha_1 \underline{b}_1^2 + w_3 \sqrt{\alpha_1} \underline{b}_1.$$

The intervention problem reads

$$\begin{aligned} \max_{\underline{\boldsymbol{b}}} \quad & w_1 \underline{\boldsymbol{a}}^{*\mathsf{T}} \underline{\boldsymbol{a}}^* + w_2 \alpha_1 \underline{b}_1^2 + w_3 \sqrt{\alpha_1} \underline{b}_1 \\ \text{subject to} \quad & \underline{a}_\ell^* = \sqrt{\alpha_\ell} \underline{b}_\ell \\ & \sum_\ell (\underline{b}_\ell - \hat{\underline{b}}_\ell)^2 \leq C. \end{aligned}$$

Using the expression for equilibrium actions, we obtain:

$$\begin{aligned} \max_{\underline{\boldsymbol{b}}} \quad & w_1 \sum_{\ell=1} \alpha_\ell \underline{b}_\ell^2 + w_2 \alpha_1 \underline{b}_1^2 + w_3 \sqrt{\alpha_1} \underline{b}_1 \\ \text{subject to} \quad & \sum_\ell (\underline{b}_\ell - \hat{\underline{b}}_\ell)^2 \leq C. \end{aligned}$$

Recalling the definition $x_\ell = \frac{\underline{b}_\ell - \hat{\underline{b}}_\ell}{\hat{\underline{b}}_\ell}$ for every ℓ, we finally rewrite the problem as:

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & w_1 \sum_{\ell=1} \alpha_\ell \hat{\underline{b}}_\ell^2 (1 + x_\ell)^2 + w_2 \alpha_1 \hat{\underline{b}}_1^2 (1 + x_1)^2 + w_3 \sqrt{\alpha_1} \hat{\underline{b}}_1 (1 + x_1) \\ \text{subject to} \quad & \sum_\ell \hat{\underline{b}}_\ell^2 x_\ell^2 \leq C. \end{aligned}$$

Theorem OA1 characterizes optimal interventions for two cases: (i) $w_1 \geq 0$ and (ii) $w_1 < 0$ and $\sum_{\ell=2} \hat{\underline{b}}_\ell^2 > C$. The extension of the analysis for the remaining case $w_1 < 0$ and $\sum_{\ell=2} \hat{\underline{b}}_\ell^2 < C$ is explained in Remark OA1, which is presented after the proof of Theorem OA1. Taken together, Theorem OA1, and Remark OA1 following it, constitute our extension of Theorem 1 to games that do not satisfy Property A.

Theorem OA1. Suppose Assumptions 1, 2 and OA1 hold. Suppose that either: (i) $w_1 \geq 0$ or that (ii) $w_1 < 0$ and $\sum_{\ell=2} \hat{\underline{b}}_\ell^2 > C$. The optimal intervention is characterized as follows:

1. $$x_1^* = \frac{\alpha_1}{\mu - (w_1 + w_2)\alpha_1} \left[w_1 + w_2 + \frac{w_3}{2\sqrt{\alpha_1} \hat{\underline{b}}_1} \right],$$

   and, for all $\ell \geq 2$,

   $$x_\ell^* = \frac{w_1 \alpha_\ell}{\mu - w_1 \alpha_\ell}.$$

   The shadow price of the planner's budget, $\mu > (w_1 + w_2)\alpha_1$, is uniquely determined as the solution of:

   $$\sum_{\ell=2} \hat{\underline{b}}_\ell^2 \left(\frac{w_1 \alpha_\ell}{\mu - w_1 \alpha_\ell} \right)^2 + \hat{\underline{b}}_1^2 \left(\frac{\alpha_1}{\mu - (w_1 + w_2)\alpha_1} \right)^2 \left[w_1 + w_2 + \frac{w_3}{2\sqrt{\alpha_1} \hat{\underline{b}}_1} \right]^2 = C$$

2. a. For all $\ell \neq 1$, $x_\ell^* > 0$ if and only if $w_1 > 0$;

b. $x_1^* > 0$ if and only if $w_1 + w_2 + \frac{w_3}{2\sqrt{\alpha_1}\hat{\underline{b}}_1} > 0$. 2b. If the game has strategic complements, $\beta > 0$, then $|x_2^*| > |x_3^*| > \cdots > |x_n^*|$. If the game has strategic substitutes, $\beta < 0$, then $|x_2^*| < |x_3^*| < \cdots < |x_n^*|$.

3. Suppose $w_1 \neq 0$. In the limit as $C \to 0$, $\mu \to \infty$ and:

$$\begin{aligned} \frac{x_\ell^*}{x_{\ell'}^*} &\to \frac{\alpha_\ell}{\alpha_{\ell'}} \text{ for all } \ell, \ell' \neq 1 \\ \frac{x_1^*}{x_\ell^*} &\to \frac{\alpha_1}{\alpha_\ell}\left[w_1 + w_2 + \frac{w_3}{2\sqrt{\alpha_1}\hat{\underline{b}}_1}\right] \text{ for all } \ell \neq 1 \end{aligned}$$

4. Suppose the game has strategic complements, $\beta > 0$. In the limit as $C \to \infty$, $\mu \to \max\{w_1\alpha_2, (w_1 + w_2)\alpha_1\}$, and
   a. If $w_1\alpha_2 > (w_1 + w_2)\alpha_1$ then

$$\begin{aligned} x_1^* &\to \frac{\alpha_1}{w_1\alpha_2 - (w_1 + w_2)\alpha_1}\left[w_1 + w_2 + \frac{w_3}{2\hat{\underline{b}}_1\sqrt{\alpha_1}}\right], \\ |x_2^*| &\to \infty, \\ |x_\ell^*| &\to \frac{\alpha_\ell}{\alpha_2 - \alpha_\ell} \text{ for all } \ell > 2. \end{aligned}$$

   b. If $w_1\alpha_2 < (w_1 + w_2)\alpha_1$ then

$$\begin{aligned} |x_1^*| &\to \infty \\ x_\ell^* &\to \frac{w_1\alpha_\ell}{(w_1 + w_2)\alpha_1 - w_1\alpha_\ell} \text{ for all } \ell \geq 2. \end{aligned}$$

5. Suppose the game has strategic substitutes, $\beta < 0$. In the limit as $C \to \infty$, $\mu \to \max\{w_1\alpha_n, (w_1 + w_2)\alpha_1\}$. Hence:
   a. If $w_1\alpha_n > (w_1 + w_2)\alpha_1$ then:

$$\begin{aligned} x_1^* &\to \frac{\alpha_1}{w_1\alpha_n - (w_1 + w_2)\alpha_1}\left[w_1 + w_2 + \frac{w_3}{2\hat{\underline{b}}_1\sqrt{\alpha_1}}\right], \\ |x_\ell^*| &\to \frac{\alpha_\ell}{\alpha_n - \alpha_\ell} \text{ for all } \ell \in \{2, \ldots, n-1\}, \\ |x_n^*| &\to \infty. \end{aligned}$$

   b. If $w_1\alpha_n < (w_1 + w_2)\alpha_1$ then

$$\begin{aligned} |x_1^*| &\to \infty \\ x_\ell^* &\to \frac{w_1\alpha_\ell}{(w_1 + w_2)\alpha_1 - w_1\alpha_\ell} \text{ for all } \ell \geq 2. \end{aligned}$$

Before the proof, we briefly explain the sense in which this extends Theorem 1 and associated results in the basic model. The formula for x_ℓ^* in part 1 is a direct generalization of equation (5), with the shadow price characterized by an equation analogous to (6). The monotonicity relations on x_ℓ^* in part 2 correspond to Corollary 1. The small-C analysis of part 3 corresponds to Proposition 1. The large-C analysis in parts 4 and 5 corresponds to the limits studied in Section 4.2.

Proof of Theorem OA1. **Part 1.** For a given $\boldsymbol{x} \in \mathbb{R}^n$, define

$$\begin{aligned} K(x_1) &= (w_1 + w_2)\alpha_1 \hat{\underline{b}}_1^2 (1 + x_1)^2 + w_3 \sqrt{\alpha_1} \hat{\underline{b}}_1 (1 + x_1) \\ C(x_1) &= C - \hat{\underline{b}}_1^2 x_1^2. \end{aligned}$$

The maximization problem can be rewritten as:

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & w_1 \sum_{\ell=2} \alpha_\ell \hat{\underline{b}}_\ell^2 (1 + x_\ell)^2 + K(x_1) \\ \text{subject to} \quad & \sum_{\ell=2} \hat{\underline{b}}_\ell^2 x_\ell^2 \leq C(x_1) \end{aligned}$$

We solve this problem in two steps.

First Step. We fix x_1 so that $C(x_1) \geq 0$; that is, $x_1 \in [-C/\hat{\underline{b}}_1, C/\hat{\underline{b}}_1]$. We then solve

$$\begin{aligned} \max_{\boldsymbol{x}_{-1}} \quad & w_1 \sum_{\ell=2} \alpha_\ell \hat{\underline{b}}_\ell^2 (1 + x_\ell)^2 \\ \text{subject to} \quad & \sum_{\ell=2} \hat{\underline{b}}_\ell^2 x_\ell^2 \leq C(x_1) \end{aligned}$$

In the case in which $w_1 = 0$ we skip this first step. If $w_1 \neq 0$, then we argue in a way exactly analogous to the proof of Theorem 1 that for all $\ell \neq 1$,

$$x_\ell^* = \frac{w_1 \alpha_\ell}{\mu - w_1 \alpha_\ell}$$

where, for all $\ell \neq 1$, $\mu \geq w_1 \alpha_\ell$ and it solves

$$\sum_{\ell=2} \hat{\underline{b}}_\ell^2 \left(\frac{w_1 \alpha_\ell}{\mu - w_1 \alpha_\ell} \right)^2 = C(x_1).$$

Note that, for all $\ell \geq 2$, $x_\ell^* > 0$ if $w_1 > 0$ and $x_\ell^* < 0$ if $w_1 < 0$.

Note also that if $w_1 < 0$ the constraint binds: the bliss point ($x_\ell^* = -1$ for all $\ell \neq 1$) cannot be achieved because $C < \sum_{\ell=2}^n \hat{\underline{b}}_\ell^2$.

Second Step. Substituting into the objective function the expression for x_ℓ^*, for all $\ell \geq 2$, we obtain:

$$\begin{aligned} \max_{x_1} \quad & W = w_1 \sum_{\ell=2} \alpha_\ell \hat{\underline{b}}_\ell^2 \left(\frac{\mu}{\mu - w_1 \alpha_\ell} \right)^2 + K(x_1) \\ \text{subject to} \quad & \sum_{\ell=2} \hat{\underline{b}}_\ell^2 \left(\frac{w_1 \alpha_\ell}{\mu - w_1 \alpha_\ell} \right)^2 = C(x_1) \\ & x_1 \in \left[-\frac{C}{\hat{\underline{b}}_1}, \frac{C}{\hat{\underline{b}}_1} \right] \end{aligned}$$

The following lemma is instrumental to the solution of this problem. It characterizes μ, which is implicitly a function of x_1.

LEMMA **OA2.** From the budget constraint in the above problem it follows that

1. $\lim_{x_1 \to -\sqrt{C}/\hat{\underline{b}}_1} \mu = \lim_{x_1 \to \sqrt{C}/\hat{\underline{b}}_1} \mu = \infty$

2.
$$\frac{d\mu}{dx_1} = \frac{\hat{\underline{b}}_1^2 x_1}{\sum_{\ell=2} \frac{w_1^2 \hat{\underline{b}}_1^2 \alpha_\ell^2}{(\mu - w_1\alpha_\ell)^3}}$$

3. $\frac{d\mu}{dx_1} > 0$ if $x_1 > 0$ and $\frac{d\mu}{dx_1} < 0$ if $x_1 < 0$;
4. $\lim_{x_1 \to -\sqrt{C}/\hat{\underline{b}}_1} \frac{d\mu}{dx_1} = -\infty$ and $\lim_{x_1 \to \sqrt{C}/\hat{\underline{b}}_1} \frac{d\mu}{dx_1} = \infty$.

Proof of Lemma OA2. The proof of part 1 of Lemma OA2 follows directly by inspection of the budget constraint. Expression 2 in part 2 of Lemma OA2 is derived by implicit differentiation of the budget constraint. Part 3 and part 4 of Lemma OA2 follow by inspection of the expression in part 2, and the fact that $\mu > w_1\alpha_\ell$. This concludes the proof of Lemma OA2. □

Lemma OA2 implies that μ as a function of $x_1 \in \left[-C/\hat{\underline{b}}_1, C\hat{\underline{b}}_1\right]$ is U-shaped; the slope is $-\infty$ at $x_1 = -C/\hat{\underline{b}}_1$ and $+\infty$ at $x_1 = C/\hat{\underline{b}}_1$; and it reaches a minimum at $x_1 = 0$.

For $w_1 \neq 0$, taking the derivative of the objective function W in expression (OA-2) with respect to x_1, we obtain:

$$\frac{dW}{dx_1} = -2\mu \sum_{\ell=2} \frac{w_1^2 \hat{\underline{b}}_1^2 \alpha_\ell^2}{(\mu - w_1\alpha_\ell)^3} \frac{d\mu}{dx_1} + 2(w_1 + w_2)\alpha_1 \hat{\underline{b}}_1^2 (1 + x_1) + w_3\sqrt{\alpha_1}\hat{\underline{b}}_1.$$

Plugging in expression for $\frac{d\mu}{dx_1}$ in part 2 of Lemma OA2 we obtain that:

$$\frac{dW}{dx_1} = -2\mu \hat{\underline{b}}_1^2 x_1 + 2(w_1 + w_2)\alpha_1 \hat{\underline{b}}_1^2 (1 + x_1) + w_3\sqrt{\alpha_1}\hat{\underline{b}}_1.$$

Part 1 of Lemma OA2 implies that $\frac{dW}{dx_1} \to \infty$ when $x_1 \to -\sqrt{C}/\hat{\underline{b}}_1$, whereas $\frac{dW}{dx_1} \to -\infty$ when $x_1 \to \sqrt{C}/\hat{\underline{b}}_1$. Hence, the optimal x_1 must be interior, which implies that $\frac{dW}{dx_1} = 0$ or, equivalently:

$$x_1^* = \frac{\alpha_1}{\mu - (w_1 + w_2)\alpha_1} \left[w_1 + w_2 + \frac{w_3}{2\sqrt{\alpha_1}\hat{\underline{b}}_1} \right].$$

Substituting x_1^*, in the budget constraint

$$\sum_{\ell=2} \hat{\underline{b}}_\ell^2 \left(\frac{w_1\alpha_\ell}{\mu - w_1\alpha_\ell} \right)^2 = C(x_1^*),$$

we obtain that the Lagrange multiplier μ must solve:

$$\sum_{\ell=2} \hat{\underline{b}}_\ell^2 \left(\frac{w_1\alpha_\ell}{\mu - w_1\alpha_\ell} \right)^2 + \hat{\underline{b}}_1^2 \left(\frac{\alpha_1}{\mu - (w_1 + w_2)\alpha_1} \right)^2 \left[w_1 + w_2 + \frac{w_3}{2\sqrt{\alpha_1}\hat{\underline{b}}_1} \right]^2 = C.$$

The conclusion for $w_1 = 0$ are obtained by taking the limits as $w_1 \to 0$ of the expression x_1^* and the expression determining μ. This concludes the proof of part 1 of Theorem OA1.

Part 2. We have already proved that, for all $\ell \geq 2$, $x_\ell^* > 0$ if and only if $w_1 > 0$. We now claim that $x_1^* > 0$ if and only if $w_1 + w_2 + \frac{w_3}{2\hat{\underline{b}}_1\sqrt{\alpha_1}} > 0$. Suppose, toward a contradiction, that

$x_1^* < 0$. Suppose, toward a contradiction, that $x_1^* < 0$. By inspection of the maximization problem

$$\begin{aligned} \max_{\underline{\boldsymbol{x}}} \quad & w_1 \sum_{\ell=2} \alpha_\ell \hat{\underline{b}}_\ell^2 (1 + x_\ell)^2 + K(x_1) \\ \text{subject to} \quad & \sum_{\ell=2} \hat{\underline{b}}_\ell^2 x_\ell^2 \leq C(x_1) \end{aligned}$$

note that if $w_1 + w_2 + \frac{w_3}{2\hat{\underline{b}}_1\sqrt{\alpha_1}} > 0$ and $x_1^* < 0$, then, by flipping the sign of x_1^*, $K(x_1)$ increases and the constraint is unaltered; this is a contradiction to our initial assumption that x_1^* was optimal.

We have just established that $x_1^* > 0$. Now, by (OA3.1) above, $x_1^* > 0$ if and only if $w_1 + w_2 + \frac{w_3}{2\hat{\underline{b}}_1\sqrt{\alpha_1}} > 0$. And since

$$x_1^* = \frac{\alpha_1}{\mu - (w_1 + w_2)\alpha_1} \left[w_1 + w_2 + \frac{w_3}{2\sqrt{\alpha_1}\hat{\underline{b}}_1} \right]$$

it follows that $\mu > \alpha_1(w_1 + w_2)$. Finally, if the game has strategic complements then $\alpha_2 > \cdots > \alpha_n$ and so $|x_2^*| > |x_3^*| > \cdots > |x_n^*|$, and if the game has strategic substitutes then $\alpha_2 < \cdots < \alpha_n$ and so $|x_2^*| < |x_3^*| < \cdots < |x_n^*|$.

Part 3. This follows by using the characterization in part 1 and by noticing that if $C \to 0$ then $\mu \to \infty$.

Part 4 and Part 5. Both parts follow by using the characterization together with the following fact, which we will now establish.

$$\lim_{C \to \infty} \mu = \max\{w_1 \max\{\alpha_2, \alpha_n\}, (w_1 + w_2)\alpha_1\}.$$

To show this, recall from above that we have the following equation for the Lagrange multiplier:

$$\sum_{\ell=2} \hat{\underline{b}}_\ell^2 \left(\frac{w_1 \alpha_\ell}{\mu - w_1 \alpha_\ell} \right)^2 + \hat{\underline{b}}_1^2 \left(\frac{\alpha_1}{\mu - (w_1 + w_2)\alpha_1} \right)^2 \left[w_1 + w_2 + \frac{w_3}{2\sqrt{\alpha_1}\hat{\underline{b}}_1} \right]^2 = C$$

If C tends to ∞ it must be that either the first denominator $(\mu - w_1\alpha_\ell)$ or the second denominator $(\mu - (w_1 + w_2)\alpha_1)$ tends to zero. Concerning the first one, this is true if either $w_1\alpha_2$ or $w_1\alpha_n$ (depending on which one is positive) approaches μ. The second denominator tends to 0 if $(w_1 + w_2)\alpha_1$ tends to μ. Both denominators are positive by definition of the Lagrange multiplier, so it will be the greater of $w_1 \max\{\alpha_2, \alpha_n\}$ and $(w_1 + w_2)\alpha_1$ which tends to μ. This concludes the proof of Theorem OA1. □

A special case of Theorem OA1 is one where the planner wants to maximize the sum of equilibrium actions. This occurs when $w_1 = w_2 = 0$. In this case we obtain

Corollary **OA1.** Suppose Assumption 1, 2 and OA1 hold. Suppose that $w_1 = w_2 = 0$ and $w_3 > 0$, i.e., the planner wants to maximize the sum of equilibrium actions. Then the optimal intervention is $\boldsymbol{b}^* = \hat{\boldsymbol{b}} + \boldsymbol{u}^1\sqrt{C}$.

Remark **OA1.** Suppose $w_1 < 0$ and $\sum_{\ell=2} \underline{b}_\ell^2 < C$, in contrast to what was assumed in the theorem. If x_1 is sufficiently small, the solution in Step 1 in the proof of Theorem OA1 entails $x_\ell = -1$ for all $\ell \geq 2$. That is, fixing x_1, the bliss point can be achieved with the remaining budget after the cost of implementing x_1, namely $C(x_1)$, is paid. Thus, when we move to Step 2 and optimize over x_1, we need to take into account that, for small values of x_1, Step 1 yields a corner solution. Hence, the analysis of how the network multiplier changes when x_1 changes will need to be adapted accordingly.

Example OA1, continued. Social interaction and peer effects

We conclude this section by applying Theorem OA1 to Example OA1 from Online Appendix Section OA3.1. In this example $w_1 = 1$, $w_2 = 0$ and $w_3 = -\gamma\sqrt{n}(n-1)$.

Corollary **OA2.** The optimal intervention in Example OA1 is characterized by

$$x_1^* = \frac{\alpha_1}{\mu - \alpha_1}\left[1 - \gamma\frac{\sqrt{n}(n-1)}{2\sqrt{\alpha_1}\hat{\underline{b}}_1}\right]$$

and, for all $\ell \geq 2$:

$$x_\ell^* = \frac{\alpha_\ell}{\mu - \alpha_\ell}$$

where the Lagrange multiplier μ solves

$$\sum_{\ell=2}\hat{\underline{b}}_\ell^2\left(\frac{\alpha_\ell}{\mu-\alpha_\ell}\right)^2 + \hat{\underline{b}}_1^2\left(\frac{\alpha_1}{\mu-\alpha_1}\right)^2\left[1 - \gamma\frac{\sqrt{n}(n-1)}{2\sqrt{\alpha_1}\hat{\underline{b}}_1}\right]^2 = C.$$

Corollary **OA3.** Consider the optimal intervention in Example OA1. It has the following properties.

1. $x_2^* > \cdots > x_n^* > 0$; $x_1^* > 0$ if and only $\gamma < \frac{2\sqrt{\alpha_1}\hat{\underline{b}}_1}{\sqrt{n}(n-1)}$
2. If $C \to 0$
$$\frac{x_\ell^*}{x_{\ell'}^*} \to \frac{\alpha_\ell}{\alpha_{\ell'}}, \text{ for all } \ell, \ell' \neq 1$$
$$\frac{x_1^*}{x_\ell^*} \to \frac{\alpha_1}{\alpha_\ell}\left[1 - \gamma\frac{\sqrt{n}(n-1)}{2\sqrt{\alpha_1}\hat{\underline{b}}_1}\right], \text{ for all } \ell \neq 1$$
3. If $C \to \infty$ then $|x_1^*| \to \infty$ and $x_\ell^* \to \frac{\alpha_\ell}{(\alpha_1 - \alpha_\ell)}$ for all $\ell \geq 2$.

OA3.2. **Beyond symmetric and non-negative $\boldsymbol{G}$.** In this subsection we relax the assumption that $\boldsymbol{G}$ is symmetric. Recall that equilibrium actions are determined by:

$$\boldsymbol{a}^* = [\boldsymbol{I} - \beta\boldsymbol{G}]^{-1}\boldsymbol{b}.$$

When $\boldsymbol{G}$ is not symmetric, we employ the *singular value decomposition* (SVD) of the matrix $\boldsymbol{M} = \boldsymbol{I} - \beta\boldsymbol{G}$. This allows us to obtain an orthogonal decomposition of an intervention useful for examining welfare, analogous to the diagonalization. An SVD of $\boldsymbol{M}$ is defined to be a tuple $(\boldsymbol{U}, \boldsymbol{S}, \boldsymbol{V})$ satisfying:

$$\boldsymbol{M} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^\mathsf{T}, \tag{OA-2}$$

where:

(1) $\boldsymbol{U}$ is an orthogonal $n \times n$ matrix whose columns are eigenvectors of $\boldsymbol{M}\boldsymbol{M}^\mathsf{T}$;
(2) $\boldsymbol{V}$ is an orthogonal $n \times n$ matrix whose columns are eigenvectors of $\boldsymbol{M}^\mathsf{T}\boldsymbol{M}$;
(3) $\boldsymbol{S}$ is an $n \times n$ matrix with all off-diagonal entries equal to zero and nonnegative diagonal entries $S_{ll} = s_l$, which are called *singular values* of $\boldsymbol{M}$. As a convention, we order the singular values so that $s_\ell > s_{\ell+1}$.

It is a standard fact that an SVD exists.[7] For expositions of the SVD, see Golub and Van Loan (1996) and Horn and Johnson (2012). The ℓ^{th} left singular vector of $\boldsymbol{M}$ corresponds to the ℓ^{th} principal component of $\boldsymbol{M}$. When $\boldsymbol{G}$ is symmetric, the SVD of $\boldsymbol{M} = \boldsymbol{I} - \beta\boldsymbol{G}$ can be taken to have $\boldsymbol{U} = \boldsymbol{V}$, and the SVD basis is one in which $\boldsymbol{G}$ is diagonal.

Let $\underline{\boldsymbol{a}} = \boldsymbol{V}^{\mathsf{T}}\boldsymbol{a}$ and $\underline{\boldsymbol{b}} = \boldsymbol{U}^{\mathsf{T}}\boldsymbol{b}$; then the equilibrium condition implies that:

$$\underline{a}_\ell^* = \frac{1}{s_\ell}\underline{b}_\ell^2,$$

and therefore the objective function is:

$$W(\boldsymbol{b}, \boldsymbol{G}) = w\left(\boldsymbol{a}^*\right)^{\mathsf{T}}\boldsymbol{a}^* = w\underline{\boldsymbol{a}}^{*\mathsf{T}}\underline{\boldsymbol{a}}^*.$$

It is now apparent that the analysis of the optimal intervention can be carried out in the same way as in Section 4. Theorem 1 applies, with the only difference that now $\alpha_\ell = 1/s_\ell^2$. We can also extend Proposition 1 and Proposition 2. As the budget tends to 0, $r_\ell^*/r_{\ell'}^*$ tends to $\alpha_\ell/\alpha_{\ell'}$; on the other hand, when C is very large, the optimal intervention is proportional to the first principal component of $\boldsymbol{M}$, and a simple intervention that focuses on the first principal component performs (nearly) as well as the optimal intervention. When $\boldsymbol{G}$ is symmetric, the nature of strategic interactions (determined by β) pins down the principal component that most amplifies an intervention. If $\boldsymbol{G}$ is non-symmetric, the singular values s_l of $\boldsymbol{M}$ are not equal to $1 - \beta\lambda_l$, where λ_l are the eigenvalues of $\boldsymbol{G}$; the singular vectors of $\boldsymbol{M}$ are not the eigenvectors of $\boldsymbol{G}$; and the left and right singular vectors need not be the same.

OA3.3. **More general costs of intervention.** In Section 4 we solved the optimal intervention problem under a specific cost function. This section discusses some natural properties on a cost function. We then show that our analysis of the optimal intervention extends to the general class of cost functions defined by these properties, as long as the budget is small.

We begin by developing properties that a reasonable cost function $(\boldsymbol{b}, \hat{\boldsymbol{b}}) \mapsto K(\boldsymbol{b}; \hat{\boldsymbol{b}})$ must satisfy.

Assumption OA2.

(1) *Translation-invariance:* For any $\boldsymbol{z} \in \mathbb{R}^n$, we have $K(\boldsymbol{b} + \boldsymbol{z}; \hat{\boldsymbol{b}} + \hat{\boldsymbol{z}}) = K(\boldsymbol{b}; \hat{\boldsymbol{b}})$, that is., there is a function $\kappa : \mathbb{R}^n \to \mathbb{R}$ such that $K(\boldsymbol{b}; \hat{\boldsymbol{b}}) = \kappa(\boldsymbol{b} - \hat{\boldsymbol{b}})$.
(2) *Symmetry:* For any permutation σ of $\{1, \ldots, n\}$, it is true that $\kappa(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}) = \kappa(y_1, y_2, \ldots, y_n)$.
(3) *Nonnegativity:* κ is nonnegative, and $\kappa(\boldsymbol{0}) = 0$.
(4) *Local separability:* $\frac{\partial^2\kappa(\boldsymbol{y})}{\partial y_i \partial y_j} = 0$ evaluated at $\boldsymbol{0}$.
(5) *Well-behaved second derivative at 0:* κ is twice differentiable with $\frac{\partial^2\kappa}{\partial y_i^2}(\boldsymbol{0}) > 0$ for all i.

Translational invariance says that there is no dependence on the starting point. Symmetry across players implies that names don't matter for costs. Nonnegativity implies that the planner cannot extract money from the system: $\kappa(\boldsymbol{0}) = 0$ is the definition of the status quo $\hat{\boldsymbol{b}}$: it does not cost anything to enact $\hat{\boldsymbol{b}}$. Local separability across individuals requires that there are no spillovers in the *costs* of interventions. This is reasonable, as it ensures

[7] The decomposition is uniquely determined up to a permutation that (i) reorders the singular values of $\boldsymbol{M}$ and correspondingly reorders the columns of $\boldsymbol{U}$ and $\boldsymbol{V}$, and (ii) flips the sign of any column of $\boldsymbol{U}$ and $\boldsymbol{V}$.

that the complementarities we study come from the benefits side and not from the costs of interventions. Finally, the twice-differentiability of the function is a technical assumption to facilitate the analysis, while the positive value of the second derivative at 0 rules out cost functions such as $\kappa(\boldsymbol{y}) = \sum_i y_i^4$ in which the increase in marginal costs at 0 is too slow.

Consider a cost function that satisfies Assumption OA2: $\kappa(\boldsymbol{y}) = \sum_i \tilde{\kappa}(y_i)$, where $\tilde{\kappa}(y) = y^2 + cy^3e^y + c'y^4$, with c and c' being arbitrary constants. Our main result is that the structure of interventions identified in Section 3.1 carries over to such cost functions as long as the budget is small.

Proposition **OA1.** Consider the intervention problem (IT) with the modification that the cost function satisfies Assumption OA2. Suppose Assumptions 1 and 2 hold and the network game satisfies Property A. At the optimal intervention, if $C \to 0$ we have $\frac{r^*_\ell}{r^*_{\ell'}} \to \frac{\alpha_\ell}{\alpha_{\ell'}}$.

Proof of Proposition OA1. First, we state and prove a lemma.

Lemma **OA3.** Under the conditions of Assumption OA2, on any compact set the function $C^{-1}\kappa(C^{1/2}\boldsymbol{z})$ converges uniformly to $k\|\boldsymbol{z}\|^2$, as $C \downarrow 0$, where $k > 0$ is some constant. We call the limit G.

Proof. Consider the Taylor expansion of κ around $\mathbf{0}$ (κ is defined by part (1) of the assumption). We will now study its properties under parts (2) to (5) of Assumption OA2. (5) ensures that the Taylor expansion exists. Local separability (4) says that there are no terms of the form y_iy_j. Non-negativity (3) (κ is nonnegative and $\kappa(\mathbf{0}) = 0$) implies that all first-order terms are zero. Also, (5) says that terms of the form y_i^2 must have positive coefficients, and symmetry (2) says that their coefficients must all be the same. $\square$

Write $\boldsymbol{y} := \boldsymbol{b} - \hat{\boldsymbol{b}}$. Let $\Delta(\boldsymbol{y})$ denote the change in welfare from the status quo. Fix all parameters of the problem, and recall the main optimization problem:

$$\max_{\boldsymbol{b}} \ \Delta(\boldsymbol{y}) \qquad \text{s.t. } \kappa(\boldsymbol{y}) \leq C \tag{IT(C)}$$

We maintain, but do not explicitly write, that welfare is evaluated at $\boldsymbol{a}^*(\boldsymbol{y})$, where $\boldsymbol{a}^* = [\boldsymbol{I} - \beta\boldsymbol{G}]^{-1}(\hat{\boldsymbol{b}} + \boldsymbol{y})$.

Let $\boldsymbol{y}(C)$ be the solution of problem IT(C), which is unique for small enough C. Then we claim that, as $C \downarrow 0$, we have

$$\frac{r^*_\ell}{r^*_{\ell'}} \to \frac{\alpha_\ell}{\alpha_{\ell'}},$$

where the similarity ratios are defined at the optimum $\boldsymbol{y}(C)$.

We will prove the result by studying an equivalent problem using Berge's Theorem of the Maximum. Let $\check{\boldsymbol{y}} = C^{-1/2}\boldsymbol{y}$. We will now define a re-scaled version of the problem, $\check{\text{IT}}(C)$.

$$\max_{\boldsymbol{b}} \ C^{-1}\Delta(C^{1/2}\check{\boldsymbol{y}}) \qquad \text{s.t. } C^{-1}\kappa(C^{1/2}\check{\boldsymbol{y}}) \leq 1. \tag{$\check{\text{IT}}(C)$}$$

This is clearly equivalent to the original problem. Let $\check{\boldsymbol{y}}^*(C)$ be the (possibly set-valued) solution for C.

The problem $\check{\text{IT}}(C)$ is not yet defined at $C = 0$, but we now define it there. Let the objective at $C = 0$ be the limit of $C^{-1}\Delta(C^{1/2}\check{\boldsymbol{y}})$ as $C \downarrow 0$, which we call F. Let the constraint be $G(\check{\boldsymbol{y}}) \leq 1$, where G is from Lemma OA3.

Let us restrict $\check{\text{IT}}(C)$ to a compact set $\mathcal{K}$ such that the constraint set $\{\boldsymbol{y} : C^{-1}\kappa(C^{1/2}\check{\boldsymbol{y}}) \leq 1\}$ is contained in $\mathcal{K}$ for all small enough C. Now we claim that the conditions of Berge's Theorem of the Maximum are satisfied: The constraint correspondence is continuous at $C = 0$ because $C^{-1}\kappa(C^{1/2}\check{\boldsymbol{y}})$ converges uniformly to G, while the objective function is jointly continuous in its two arguments.

The Theorem of the Maximum therefore implies that the maximized value is continuous at $C = 0$. Because the convergence of the objective is actually uniform on $\mathcal{K}$ by the Lemma, this is possible if and only if $\check{\boldsymbol{y}}$ approaches the solution of the problem

$$\max_{\boldsymbol{b}} \; F(\check{\boldsymbol{y}}) \quad \text{s.t. } \|\check{\boldsymbol{y}}\|^2 \leq 1.$$

By the same argument, the same point is the limit of the solutions to

$$\max_{\boldsymbol{b}} \; C^{-1}\Delta(C^{1/2}\check{\boldsymbol{y}}) \quad \text{s.t. } \|\check{\boldsymbol{y}}\|^2 \leq 1.$$

By Proposition 1, in that limit this satisfies

$$\frac{r^*_\ell}{r^*_{\ell'}} \to \frac{\alpha_\ell}{\alpha_{\ell'}}.$$

□

We next impose an additional restriction on the structure of the costs of intervention and we show that this new restriction together with Assumption OA2 fully characterizes the cost functions that we used in our main analysis.

Assumption OA3. There is a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ so that $\kappa(s\boldsymbol{y}) = f(s)\kappa(\boldsymbol{y})$.

Proposition **OA2.** Consider a cost function that satisfies Assumptions OA2 and OA3. There is a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\kappa(\boldsymbol{y}) = f(\|\boldsymbol{y}\|).$$

Proposition OA2 implies that the cost of intervention $\boldsymbol{y}$ is the same as the cost of an intervention obtained as an orthogonal transformation of $\boldsymbol{y}$; that is $\kappa(\boldsymbol{y}) = \kappa(\boldsymbol{O}\boldsymbol{y})$ with $\boldsymbol{O}$ be an orthogonal matrix. This allows to rewrite the intervention problem using the orthogonal decomposition of welfare and costs that we employ in Section 4, and all the results developed there extend to this more general environment.

We conclude by taking up the implication of linear costs of intervention. The main result is that with a linear cost function, that is, $K(\boldsymbol{b}, \hat{\boldsymbol{b}}) = \sum_i |b_i - \hat{b}_i|$, the optimal intervention will target a single individual. For ease of exposition, we will restrict attention to Example 1. The analysis can be easily extended to general network games.

We consider the following intervention problem:

$$\begin{aligned} \max_{\boldsymbol{b}} \; & (\boldsymbol{a}^*)^{\mathsf{T}} \boldsymbol{a}^* \\ \text{s.t. } & \boldsymbol{a}^* = [\boldsymbol{I} - \beta \boldsymbol{G}]^{-1}\boldsymbol{b}, \\ K(\boldsymbol{b}; \hat{\boldsymbol{b}}) = & \sum_{i \in \mathcal{N}} |b_i - \hat{b}_i| \leq C, \end{aligned} \tag{IT-Linear Cost}$$

Proposition **OA3.** The solution to problem IT-Linear Cost has the property that there exists i^* such that $b_i^* \neq \hat{b}_{i^*}$ and $b_i^* = \hat{b}_i$ for al $i \neq i^*$.

Proof of Proposition OA3. Define $W(\boldsymbol{b}) = \boldsymbol{a}(\boldsymbol{b})^\mathsf{T}\boldsymbol{a}(\boldsymbol{b})$. Let F be the set of *feasible* $\boldsymbol{b}$, those satisfying the budget constraint $K(\boldsymbol{b};\hat{\boldsymbol{b}}) \leq C$. Suppose the conclusion does not hold and let $\boldsymbol{b}^*$ be the optimum, with $W^* = W(\boldsymbol{b}^*)$. Then, because by hypothesis the optimum is not at an extreme point, F contains a line segment L such that $\boldsymbol{b}^*$ is in the interior of L.[8]

Now restrict attention to a plane P containing this L and the origin. Note that L is contained in a convex set

$$E = \{\boldsymbol{b} : W(\boldsymbol{b}) \leq W^*\}.$$

The point $\boldsymbol{b}^*$ is contained in the interior of L; thus $\boldsymbol{b}^*$ is in the interior of E. On the other hand, $\boldsymbol{b}^*$ must be on the (elliptical) boundary of E because U is strictly increasing in each component (by irreducibility of the network) and continuous. This is a contradiction. □

We now characterize the optimal target for the case of strategic complements, i.e., $\beta > 0$. Remark OA2 explains how to extend the analysis for the case of strategic substitutes.

In the case of strategic complements, it is clear that the planner uses all the budget C to increase the standalone marginal benefit of i^*, i.e., $b_i^* = \hat{b}_i + C$; reducing someone's effort can never help. Thus, the planner changes the status quo $\hat{\boldsymbol{b}}$ into $\boldsymbol{b} = \hat{\boldsymbol{b}} + C\mathbf{1}_{i^*}$ where $\mathbf{1}_{i^*}$ is a vector of 0 except for entry i^* that takes value 1. Let $\boldsymbol{a}(\mathbf{1}_i)$ be the Nash equilibrium when all individuals have $b_j = 0$ and $b_i = 1$, i.e., $\boldsymbol{a}(\mathbf{1}_i) = [\boldsymbol{I} - \beta\boldsymbol{G}]^{-1}\mathbf{1}_i$. It is easy to verify that the solution to problem IT-Linear Cost is:

$$i^* = \underset{i}{\operatorname{argmax}} \left\{ \boldsymbol{a}(\hat{\boldsymbol{b}} + C\mathbf{1}_i)^\mathsf{T}\boldsymbol{a}(\hat{\boldsymbol{b}} + C\mathbf{1}_i) - \boldsymbol{a}(\hat{\boldsymbol{b}})^\mathsf{T}\boldsymbol{a}(\hat{\boldsymbol{b}}) \right\}.$$

This is equivalent to

$$i^* = \underset{i}{\operatorname{argmax}} \left\{ C\|\boldsymbol{a}(\mathbf{1}_i)\| \left[2\|\boldsymbol{a}(\hat{\boldsymbol{b}})\|\rho(\boldsymbol{a}(\mathbf{1}_i), \boldsymbol{a}(\hat{\boldsymbol{b}})) + C\|\boldsymbol{a}(\mathbf{1}_i)\| \right] \right\}. \tag{OA-3}$$

where recall that $\rho(\boldsymbol{a}(\mathbf{1}_i), \boldsymbol{a}(\hat{\boldsymbol{b}}))$ is the cosine similarity between vectors $\boldsymbol{a}(\mathbf{1}_i)$ and $\boldsymbol{a}(\hat{\boldsymbol{b}})$. There are two characteristics of a player that determines whether the player is a good target.

The first characteristic is $\|\boldsymbol{a}(\mathbf{1}_i)\|$. This is the square root of the aggregate equilibrium utility in the game with $\boldsymbol{b} = \mathbf{1}_i$, i.e., the squared root of $\boldsymbol{a}(\mathbf{1}_i)^\mathsf{T}\boldsymbol{a}(\mathbf{1}_i)$. So, a player with a high $\|\boldsymbol{a}(\mathbf{1}_i)\|$ is a player who induces a large welfare in the game in which he is the only player with positive standalone marginal benefit. We call this the *welfare centrality* of an individual. It is convenient to express the welfare centrality of individual i in terms of principal components of $\boldsymbol{G}$. Note that

$$\|\boldsymbol{a}(\mathbf{1}_i)\| = \|\underline{\boldsymbol{a}}(\underline{\mathbf{1}}_i)\| = \sqrt{\sum_\ell \alpha_\ell (u_i^\ell)^2}.$$

Recall that under strategic complement $\alpha_1 > \alpha_2 > .. > \alpha_n$ and so an individual with a high welfare centrality is one that is highly represented in the main principal components of the network.

The second factor is $\rho(\boldsymbol{a}(\mathbf{1}_i), \boldsymbol{a}(\hat{\boldsymbol{b}}))$. This measures the vector similarity between (i) the equilibrium action profile in the game with $\boldsymbol{b} = \mathbf{1}_i$; and (ii) the status quo equilibrium action profile. A player with a large $\rho(\boldsymbol{a}(\mathbf{1}_i), \boldsymbol{a}(\hat{\boldsymbol{b}}))$ is a player that, in the game in which he is the

[8]Formally, for some $z > 0$ there is a linear map $\varphi : [-z, z] \to F$ such that $\varphi(0) = \boldsymbol{b}^*$.

only player with positive standalone marginal benefit, leads a distribution of effort similar to the distribution of effort in the status quo.

Small C. Suppose $C \approx 0$. Then the optimal target is selected based on the first term of expression (OA-3); that is:

$$i^* = \operatorname*{argmax}_i \|\boldsymbol{a}(\mathbf{1}_i)\| \rho(\boldsymbol{a}(\mathbf{1}_i), \boldsymbol{a}(\hat{\boldsymbol{b}}))$$

For small budgets, the optimal intervention focuses on the player who has a large welfare centrality and that, at the same time, leads to a distribution of effort not too different from the status quo equilibrium effort.

Large C: For C sufficiently large, the last term of expression (OA-3) dominates and therefore the player that is targeted is the player with the highest welfare centrality.

Remark **OA2** (Extension to the case of strategic substitutes)**.** In the case of strategic substitutes, we know for the targeted player i^*, $b^*_{i^*} = \hat{b}_i \pm C$, but we cannot say, a priori, which (positive or negative), and indeed it is easy to provide examples that both can happen. Under this qualification, the analysis developed for the case of strategic complements extends

OA3.4. **Intervention through monetary incentives.** In the basic model presented in Section 2, an intervention alters incentives for individual action through a direct change in marginal benefits/marginal costs. The convexity in the cost of changing these marginal benefits plays a key role in the analysis. In this section we provide a demonstration of how our approach can be applied beyond this cost setting. We do this by using our methods to solve the problem of offering monetary incentives to individuals for choosing between two actions.

Let us reinterpret a node i as a population; thus $\mathcal{N} = \{1, 2, \ldots, n\}$ is the set of populations. Within population i, there is a continuum of individuals distributed uniformly in $\mathcal{I} = [0, \overline{\tau}]$. Each individual in population i chooses whether to take action 1 or to take action 0. A strategy of an individual in population i is a function $q_i : [0, \overline{\tau}] \to [0, 1]$ that describes the probability that an individual of type $\tau_i \in [0, \overline{\tau}]$ chooses action 1. Without loss of generality, we focus on equilibria in which all the players within a population have the same strategy.

The payoff to an individual who chooses action 0 is normalized to 0. If individual τ_i takes action 1, then he incurs a cost τ_i and gets a benefit that depends on his population's standalone marginal benefit of action 1, b_i, and the number of other individuals he meets who have also taken action 1. We assume that the interaction between populations takes the form of random matching, with the following specification: An individual τ_i in population i meets someone from population j with probability g_{ij}, and, within population j, τ_i meets an individual selected uniformly at random. Suppose τ_i meets type τ_j, and let q_j be the strategy of individuals in population j. Then individual τ_i's payoff for the interaction with the random partner τ_j is

$$\widetilde{\beta} q_j(\tau_j) + b_i - \tau_i.$$

In this expression, $\widetilde{\beta} q_j(\tau_j)$ represents the payoffs from interacting with peers that have also taken action 1.

First, we show that the conditions for an equilibrium are isomorphic to those of the games we studied in Section 3.1. It is immediate to see that the best reply of each individual in population i is a cutoff strategy: there exists a cutoff $a_i \in \mathcal{I}$ so that $q(\tau_i) = 1$ for all $\tau_i \leq a_i$ and $q(\tau_i) = 0$ otherwise. The equilibrium condition for these cutoffs is that, for all $i \in \mathcal{N}$,

$$\widetilde{\beta} \sum_j g_{ij} P[\tau_j \leq a_j^*] + b_i - a_i^* = 0 \quad \Longleftrightarrow \quad a_i = b_i + \frac{\widetilde{\beta}}{\overline{\tau}} \sum_j g_{ij} a_j^*.$$

Denoting by $\beta = \widetilde{\beta}/\overline{\tau}$, the equilibrium threshold profile $\boldsymbol{a}^*$ solves

$$[\boldsymbol{I} - \beta \boldsymbol{G}]\boldsymbol{a}^* = \boldsymbol{b}.$$

The equilibrium expected payoff to group i is:

$$\begin{aligned} U_i(\boldsymbol{a}^*, \boldsymbol{b}) &= \int_0^{a_i^*} \left(\beta \sum_j g_{ij} a_j^* + b_i - \tau_i \right) d\tau_i \\ &= \int_0^{a_i^*} (a_i^* - \tau_i)\, d\tau_i = \frac{1}{2} a_i^{*2}, \end{aligned}$$

where the second equality follows by using the best response of each population. So aggregate equilibrium utility is

$$W(\boldsymbol{b}, \boldsymbol{G}) = \frac{1}{2} (\boldsymbol{a}^*)^{\mathsf{T}} \boldsymbol{a}^*.$$

Suppose the planner, before the players choose their action, commits to the a subsidy scheme. The subsidy scheme depends on realized actions, which are taken after the scheme is announced. More precisely, the planner selects a vector $\boldsymbol{y} \in \mathbb{R}^n$ and offers the following scheme:

Subsidizing action 1. If $y_i > 0$ then the planner gives a subsidy of $s_i^1(\tau_i) = \tau_i - [a_i(\boldsymbol{y}) - y_i]$ to all population i's types $\tau_i \in [a_i(\boldsymbol{y}) - y_i, a_i(\boldsymbol{y})]$ who take action 1.

Subsidizing action 0. If $y_i < 0$ then the planner gives a subsidy of $s_i^0(\tau_i) = [a_i(\boldsymbol{y}) + |y_i|] - \tau_i$ to all $\tau_i \in [a_i(\boldsymbol{y}), a_i(\boldsymbol{y}) + |y_i|]$ who do not adopt the new technology (take action 0).

We make three observations. First, under intervention $\boldsymbol{y}$ the profile of thresholds $\boldsymbol{a}(\boldsymbol{y})$ is a Nash equilibrium. Furthermore, the planner does not waste resources in the sense that she uses the minimum amount of resources to implement $\boldsymbol{a}(\boldsymbol{y})$. To see this note that, by construction, the planner provides monetary payments to take action 1 or to take action 0 only to types who need such transfers to satisfy their incentive compatibility constraint. The monetary payments make these incentive compatible constrains just bind. Finally, let $\mathbf{1}_{y_i > 0}$ be an indicator function that takes value 1 if $y_i > 0$ and 0 otherwise, then note that the cost of intervention $\boldsymbol{y}$ is

$$\begin{aligned} K(\boldsymbol{y}) &= \frac{1}{2} \sum_i \mathbf{1}_{y_i > 0} \int_{a_i(\boldsymbol{y}) - y_i}^{a_i(\boldsymbol{y})} s_i^1(\tau_i) d\tau_i + \sum_i (1 - \mathbf{1}_{y_i > 0}) \int_{a_i(\boldsymbol{y})}^{a_i(\boldsymbol{y}) + |y_i|} s_i^0(\tau_i) d\tau_i \\ &= \frac{1}{2} \sum_i y_i^2 \end{aligned}$$

We then consider a planner who intervenes in the system. The planner has complete information about the type of each individual in each population and can subsidize types to take action 1 or to take action 0, in a perfectly targeted manner. In doing this, the planner effectively shifts the b_i of some individuals in some populations. The cheapest individuals to influence are those who are close to being indifferent between the two actions, so that they do not need to be paid very much to change their behavior. Indeed, the payment to an individual is proportional to his distance x from the marginal type in equilibrium: Integrating across all the individuals whose actions are changed gives $\int_0^{y_i} x\,dx$, a cost that is quadratic in the magnitude of the change. The intervention problem turns out to be mathematically equivalent to (IT), and so all our results apply.

We can now define the intervention problem of the planner as follows. Starting from the status quo $\hat{\boldsymbol{b}}$, the planner chooses intervention $\boldsymbol{y}$ to maximize aggregate equilibrium utility under the constraint that individuals play according to equilibrium and that the cost of the intervention cannot exceed C. Formally,

$$\begin{aligned} &\max_{\boldsymbol{y}\in\mathbb{R}^n} \frac{1}{2}\boldsymbol{a}^{\mathsf{T}}\boldsymbol{a} \\ &\text{s.t. } [\boldsymbol{I}-\beta\boldsymbol{G}]\boldsymbol{a} = \hat{\boldsymbol{b}}+\boldsymbol{y}, \\ &K(\boldsymbol{y}) = \frac{1}{2}\sum_i y_i^2 \leq C, \end{aligned} \tag{IT-P}$$

Intervention problem (IT-P) is equivalent to the intervention problem (IT) defined in Section 2.

Note that the specific payoff functions we have taken here make the problem isomorphic to the setting of Example 1, but by suitably modifying the payoffs, we could capture more general externalities, along the lines of Online Appendix Section OA3.1.

We focus throughout on maximizing aggregate utility, but we note that the results have applications to other kinds of objectives, such as implementing Pareto improvements. In some cases, interventions will make everyone better off without modification, when positive externalities are strong enough to overcome any negative welfare impacts. However, even when this is not the case, the planner may be able to achieve Pareto improvements. For example, consider a planner who is able to make lump sum transfers – e.g., award or take away discretionary compensation – in addition to any targeted incentives or contingent payments. In such cases, if an improvement in aggregate utility is possible, then the planner can use such transfers to compensate individuals (for instance, those harmed by negative externalities), and achieve a Pareto improvement. In the setting discussed in this subsection, combining lump-sum and action-contingent transfers would then implement a range of Pareto improvements. Even beyond the monetary-incentives setting under consideration here, lump sum transfers may be available to the planner in addition to whatever incentive-targeting scheme is being used, and in such a setting our comments here would apply also.

References

Acemoglu, D., A. Ozdaglar, and A. Tahbaz-Salehi (2016): "Networks, Shocks, and Systemic Risk," in *Oxford Handbook of the Economics of Networks*, ed. by Y. Bramoullé, A. Galeotti, and B. Rogers, Oxford University Press.

Angeletos, G.-M. and A. Pavan (2007): "Efficient Use of Information and Social Value of Information," *Econometrica*, 75, 1103–1142.

Ballester, C., A. Calvó-Armengol, and Y. Zenou (2006): "Who's Who in Networks. Wanted: The Key Player," *Econometrica*, 74, 1403–1417.

Baqaee, D. R. and E. Farhi (2017): "The Macroeconomic Impact of Microeconomic Shocks: Beyond Hulten's Theorem," Working paper, London School of Economics.

Bramoullé, Y., R. Kranton, and M. d'Amours (2014): "Strategic Interaction and Networks," *The American Economic Review*, 104, 898–930.

Calvó-Armengol, A., J. Martí, and A. Prat (2015): "Communication and Influence," *Theoretical Economics*, 10, 649–690.

Demange, G. (2017): "Optimal Targeting Strategies in a Network under Complementarities," *Games and Economic Behaviour*, 105, 84–103.

Dessein, W., A. Galeotti, and T. Santos (2016): "Rational Inattatention and Organziational Focus," *American Economic Review*, 106, 1522–36.

Elliott, M. and B. Golub (2018): "A Network Approach to Public Goods," *Journal of Political Economy*, forthcoming.

Golub, B. and C. Lever (2010): "The Leverage of Weak Ties: How Linking Groups Affects Inequality," Working paper, Harvard University.

Golub, G. H. and C. F. Van Loan (1996): *Matrix Computations*, The Johns Hopkins University Press.

Horn, R. A. and C. R. Johnson (2012): *Matrix Analysis*, New York, NY, USA: Cambridge University Press, 2nd ed.

Jackson, M., B. W. Rogers, and Y. Zenou (2017): "The Economic Consequences of Social-Network Structure ," *Journal of Economic Literature*, 55, 49–95.

Marschak, T. and R. Radner (1972): *Economic Theory of Teams*, New Haven, USA: Cowles Foundation.

Morris, S. and H. S. Shin (2002): "Social Value of Public Information," *American Economic Review*, 92, 1521–1534.